INTRODUCTION TO ALGEBRAIC D-MODULES

PAVEL ETINGOF

Abstract. These are notes of my minicourse at the workshop "Geometry and representation theory", Vienna, January 2017.

1. Lecture 1

1.1. **Differential operators.** Let k be an algebraically closed field of characteristic zero. Let X be a smooth affine algebraic variety over k. Let $O(X)$ be the algebra of regular functions on X. Following Grothendieck, we define inductively the notion of a *differential operator of order N on X*. Namely, a differential operator of order -1 is zero, and a k-linear operator $L : O(X) \to O(X)$ is a differential operator of order $N \geq 0$ if for all $f \in O(X)$, the operator $[L, f]$ is a differential operator of order $N - 1$.

Let $D_N(X)$ denote the space of differential operators of order N. We have

$$0 = D_{-1}(X) \subset O(X) = D_0(X) \subset D_1(X) \subset ... \subset D_N(X) \subset ...$$

and $D_i(X)D_j(X) \subset D_{i+j}(X)$, which implies that the nested union $D(X) := \cup_{i\geq 0} D_i(X)$ is a filtered algebra.

Definition 1.1. $D(X)$ is called *the algebra of differential operators on X.*

Exercise 1.2. 1. One has $[D_i(X), D_j(X)] \subset D_{i+j-1}(X)$ for $i, j \geq 0$. In particular, $[,]$ makes $D_1(X)$ a Lie algebra naturally isomorphic to $\mathrm{Vect}(X) \ltimes O(X)$, where $\mathrm{Vect}(X)$ is the Lie algebra of vector fields on X.

2. Suppose $x_1, ..., x_n \in O(X)$ are regular functions such that $dx_1, ..., dx_n$ form a basis in each cotangent space. Let $\partial_1, ..., \partial_n$ be the corresponding vector fields. For $\mathbf{m} = (m_1, ..., m_n)$, let $|\mathbf{m}| := \sum_{i=1}^n m_i$ and $\partial^{\mathbf{m}} := \partial_1^{m_1} ... \partial_n^{m_n}$. Then $D_N(X)$ is a free finite rank $O(X)$-module (under left multiplication) with basis $\{\partial^{\mathbf{m}}\}$ with $|\mathbf{m}| \leq N$, and $D(X)$ is a free $O(X)$-module with basis $\{\partial^{\mathbf{m}}\}$ for all $\mathbf{m}$.

3. One has $\mathrm{gr}\, D(X) = \oplus_{i\geq 0} \Gamma(X, S^i TX) = O(T^*X)$. In particular, $D(X)$ is left and right Noetherian and has finite homological dimension ($\leq 2 \dim X$).

4. $D(X)$ is generated by $O(X)$ and elements L_v, $v \in \mathrm{Vect}(X)$ (depending linearly on v), with defining relations

$$(1) \qquad [f, g] = 0,\ [L_v, f] = v(f),\ L_{fv} = fL_v,\ [L_v, L_w] = L_{[v,w]},$$

where $f, g \in O(X), v, w \in \mathrm{Vect}(X)$.

5. If $U \subset X$ is an affine open set then the multiplication map $O(U) \otimes_{O(X)} D(X) \to D(U)$ is a filtered isomorphism.

1.2. **D-modules.**

Definition 1.3. A left (respectively, right) D-module on X is a left (respectively, right) $D(X)$-module.

Example 1.4. 1. $O(X)$ is an obvious example of a left D-module on X. Also, $\Omega(X)$ (the top differential forms on X) is naturally a right D-module, via $\rho(L) = L^*$ (the adjoint differential operator to L with respect to the "integration pairing" between functions and top forms). More precisely, $f^* = f$ for $f \in O(X)$, and L_v^* is the action of the vector field $-v$ on top forms (by Lie derivative). Finally, $D(X)$ is both a left and a right D-module on X.

2. Suppose $k = \mathbb{C}$, and f is a holomorphic function defined on some open set in X. Then $M(f) := D(X)f$ is a left D-module. We have a natural surjection $D(X) \to M(f)$ whose kernel is the left ideal generated by the linear differential equations satisfied by f. E.g. $M(1) = O(X) = D(X)/D(X)\mathrm{Vect}(X)$, $M(x^s) = D(\mathbb{C})/D(\mathbb{C})(x\partial - s)$ if $s \notin \mathbb{Z}_{\geq 0}$, $M(e^x) = D(\mathbb{C})/D(\mathbb{C})(\partial - 1)$. Similarly, if ξ is a distribution (e.g., a measure) then $\xi \cdot D(X)$ is a right D-module. For instance, $\delta \cdot D(\mathbb{C}) = D(\mathbb{C})/xD(\mathbb{C})$, where δ is the delta-function on the line.

Exercise 1.5. Show that for any nonzero regular function f on X, $M(f) = O(X)$.

1.3. **D-modules on non-affine varieties.** Now assume that X is a smooth variety which is not necessarily affine. In this case, Exercise 1.2(5) implies that there exists a canonical quasicoherent sheaf of algebras D_X on X such that $\Gamma(U, D_X) = D(U)$ for any affine open set $U \subset X$. This sheaf is called the *sheaf of differential operators on X*.

Definition 1.6. A left (respectively, right) D-module on X is a quasicoherent sheaf of left (respectively, right) D_X-modules. The categories of left (respectively, right) D-modules on X (with obviously defined morphisms) are denoted by $\mathcal{M}_l(X)$ and $\mathcal{M}_r(X)$.

Note that if X is affine, this definition is equivalent to the previous one (by taking global sections).

As before, the basic examples are O_X (a left D-module), Ω_X (a right D-module), D_X (both a left and a right D-module).

We see that the notion of a D-module on X is local. For this reason, many questions about D-modules are local and reduce to the case of affine varieties.

1.4. **Connections.** We will need an algebraic notion of a connection. In differential geometry we have a theory of connections on vector bundles. An algebraic vector bundle on X is the same thing as a coherent, locally free O_X-module. It turns out that the usual definition of a connection, when written algebraically, makes sense for any O_X-module (i.e., quasicoherent sheaf), not necessarily coherent or locally free.

Namely, let Ω_X^i be the O_X-module of differential i-forms on X.

Definition 1.7. A connection on an O_X-module M is a k-linear morphism of sheaves $\nabla : M \to M \otimes_{O_X} \Omega_X^1$ such that

$$\nabla(fm) = f\nabla(m) + m \otimes df$$

for local sections f of O_X and m of M.

Thus for each $v \in \mathrm{Vect}(X)$ we have the operator of covariant derivative $\nabla_v : M \to M$ given on local sections by $\nabla_v(m) := \nabla(m)(v)$.

Exercise 1.8. Let X be an affine variety. Show that the operator $m \mapsto ([\nabla_v, \nabla_w] - \nabla_{[v,w]})m$ is $O(X)$-linear in v, w, m.

Given a connection ∇ on M, define the O_X-linear map $\nabla^2 : M \to M \otimes_{O_X} \Omega_X^2$ given on local sections by

$$\nabla^2(m)(v, w) := ([\nabla_v, \nabla_w] - \nabla_{[v,w]})m.$$

This map is called the *curvature* of ∇. We say that ∇ is *flat* if its curvature vanishes: $\nabla^2 = 0$.

Proposition 1.9. *A left D_X-module is the same thing as an O_X-module with a flat connection.*

Proof. Given an O_X-module M with a flat connection ∇, we extend the O_X-action to a D_X-action by $\rho(L_v) = \nabla_v$. The first three relations of (1) then hold for any connection, while the last relation holds due to flatness of ∇. Conversely, the same formula can be used to define a flat connection ∇ on any D_X-module M. □

Exercise 1.10. Show that if a left D-module M on X is O-coherent (i.e. a coherent sheaf on X) then it is locally free, i.e., is a vector bundle with a flat connection, and vice versa.

1.5. **Left and right D-modules.** One might wonder if one can relate right D-modules to left D-modules. It is clear that a right D_X-module is the same thing as a left D_X^{op}-module. Moreover, even though the algebra $D(X)^{\mathrm{op}}$ is not always isomorphic to $D(X)$ for affine X, these algebras are always Morita equivalent, in fact canonically so. In other words, we have

Proposition 1.11. *There is a canonical equivalence*

$$\tau : \mathcal{M}_l(X) \cong \mathcal{M}_r(X)$$

between the categories of left and right D-modules on X.

Proof. Assume first that X is affine. To construct τ, note that $D(X)^{\mathrm{op}}$ is generated by $O(X)$ and elements R_v, $v \in \mathrm{Vect}(X)$ (depending linearly on v), with defining relations

$$(2) \qquad [f, g] = 0,\ [R_v, f] = -v(f),\ R_{fv} = R_v f,\ [R_v, R_w] = -R_{[v,w]},$$

where $f, g \in O(X)$, $v, w \in \mathrm{Vect}(X)$. Now for a left D-module M on X, set $\tau(M) := M \otimes_{O(X)} \Omega(X)$, and let

$$R_v|_{\tau(M)} = -L_v|_M \otimes 1 - 1 \otimes \mathrm{Lie}_v,$$

where the last term denotes the Lie derivative of top forms. This is well defined (check it!). It remains to check the relations. Let us check the relation $R_{fv} = R_v f$ (we leave the others as exercises). We have

$$R_v f - R_{fv} = L_{fv} \otimes 1 + 1 \otimes \mathrm{Lie}_{fv} - (L_v \otimes 1 + 1 \otimes \mathrm{Lie}_v)(1 \otimes f) = 0,$$

since $\mathrm{Lie}_{fv} = \mathrm{Lie}_v f$ on $\Omega(X)$.

Conversely, if N is a right D-module on X, then we define $\tau^{-1}(N) := N \otimes_{O(X)} \Omega(X)^{-1}$, and

$$L_v|_{\tau^{-1}(N)} := -R_v|_N \otimes 1 + 1 \otimes \mathrm{Lie}_v.$$

This construction is clearly compatible with gluing, so extends to arbitrary smooth X. $\square$

Thus, we may think of a single category $\mathcal{M}(X)$ of D-modules on X, realizing its objects as left or right D-modules, whichever is more convenient in a given situation. This is clearly an abelian category.

1.6. **Direct and inverse images.** Let $\pi : X \to Y$ be a morphism of smooth affine varieties. This morphism gives rise to a homomorphism $\pi^* : O(Y) \to O(X)$, making $O(X)$ an $O(Y)$-module, and a morphism of vector bundles $\pi_* : TX \to \pi^*TY$. This induces a map on global sections $\pi_* : \mathrm{Vect}(X) \to O(X) \otimes_{O(Y)} \mathrm{Vect}(Y)$.

Define

$$D_{X\to Y} = O(X) \otimes_{O(Y)} D(Y)$$

This is clearly a right $D(Y)$-module. Let us show that it also has a commuting left $D(X)$-action. The left action of $O(X)$ is obvious, so it remains to construct a flat connection. Given a vector field v on X, let

$$\nabla_v(f\otimes L) = v(f)\otimes L + f\pi_*(v)L,\ f\in O(X),\ L\in D(Y), \tag{3}$$

where we view $\pi_*(v)$ as an element of $D_{X\to Y}$. This is well defined since for $a\in O(Y)$ one has $[\pi_*(v),a] = v(a)\otimes 1$.

Exercise 1.12. Show that this defines a flat connection on $D_{X\to Y}$.

Now we define the *inverse image functor* $\pi^0 : \mathcal{M}_l(Y)\to\mathcal{M}_l(X)$ by

$$\pi^0(N) = D_{X\to Y}\otimes_{D(Y)} N$$

and the *direct image functor* $\pi_0 : \mathcal{M}_r(X)\to\mathcal{M}_r(Y)$ by

$$\pi_0(M) = M\otimes_{D(X)} D_{X\to Y}.$$

These functors are right exact, so we may define the derived functors $L\pi_0$ and $L\pi^0$ landing in the corresponding bounded derived categories. These functors have cohomology in nonpositive degrees. Note also that by definition, $D_{X\to Y} = \pi^0(D(Y))$.

We will denote $L\pi_0$ by π_* and call it the *full direct image functor*. Also define the (shifted) *full inverse image functor* $\pi^! = L\pi^0[d]$, with $d = \dim X - \dim Y$ for irreducible X, Y (where for a complex $K^\bullet$, we set $K[j]^i := K^{i+j}$). The usefulness of this shift will become clear below.

Note that $\pi^0(N) = O(X)\otimes_{O(Y)} N$ as an $O(X)$-module (i.e., the usual pullback of O-modules), with the connection defined by the formula similar to (3):

$$\nabla_v(f\otimes m) = v(f)\otimes m + f\nabla_{\pi_*(v)}(m),\ f\in O(X),\ m\in M.$$

This means that the definition of $\pi^!$ is local both on X and on Y. On the contrary, the definition of π_* is local only on Y but not on X (as we will see later). [1]

Thus we can use the same definition locally to define $\pi^!$ for any morphism of smooth varieties, and π_* for an affine morphism (i.e. such that $\pi^{-1}(U)$ is affine for any affine open set $U\subset Y$). We will see later that the correct functor π_* for a non-affine morphism is not the derived functor of anything and can be defined only in the derived category.

[1] This is related to the fact that the notion of inverse image of a D-module is related to the notion of the inverse image (pullback) of a function, while the notion of a direct image of a D-module is related to the notion of direct image of a measure, which involves integration over the fibers and thus is non-local along them (see Exercise 1.13).

Finally, note that the functors π_* and $\pi^!$ are compatible with compositions.

Exercise 1.13. 1. Let $\pi : X \to Y$ be a smooth morphism of smooth affine complex varieties, and f is a holomorphic function defined on some open set in Y (in the usual topology). Show that $\pi^0(M(f)) = M(\pi^*(f))$.

2. Let $Y = \mathbb{C} \setminus \{0, 1\}$, and $X = \{(t, z) \in \mathbb{C}^2 : z \neq 0, 1; t \neq 0, 1, z\}$. Let $\pi : X \to Y$ be defined by $\pi(t, z) = z$. Let $a, b, c \in \mathbb{C}$, and $M(a, b, c)$ be the D-module on X generated by the (multivalued) function

$$f(t, z) := t^a(t-1)^b(t-z)^c.$$

Show that for Weil generic a, b, c (outside of finitely many hyperplanes), $\pi_*(M(a, b, c))$ lives in degree 0 and is a rank 2 trivial vector bundle on Y with a flat connection $\nabla = d + A(z)dz$, $A \in \mathrm{Mat}_2(\mathbb{C}[z, z^{-1}, (z-1)^{-1}])$. Compute $A(z)$. Show that if the real parts of a, b are > -1 (and a, b, c are Weil generic) then $\pi_*(M(a, b, c))$ is generated by the Gauss hypergeometric function $F(z) := \int_0^1 t^a(t-1)^b(t-z)^c dt$.

2. Lecture 2

2.1. **Direct and inverse images for open embeddings.** Let $j : U \to Y$ be an open embedding of irreducible affine varieties. In this case $D_{U\to Y} = O(U) \otimes_{O(Y)} D(Y) = D(U)$, so for a D-module N on Y, $j^0(N) = O(U) \otimes_{O(Y)} N = N|_U$, the usual restriction (=localization) of N to U. In particular, $j^0 = j^!$ is an exact functor (note that there is no shift since $\dim U = \dim Y$). Since all our considerations are local, this extends verbatim to arbitrary (not necessarily affine) varieties.

Also, consider the functor j_0. We have $j_0(M) = M \otimes_{D(U)} D(U) = M$. In other words, $j_0(M)$ is simply the restriction of M to the subalgebra $D(Y) \subset D(U)$. In particular, the functor $j_0 = j_*$ is exact. Thus, $j_0 = j_*$ is exact for any affine open embedding j of not necessarily affine varieties (i.e., such that for any affine open set $V \subset Y$, the intersection $V \cap U$ is affine).

Moreover, if $j : U \to Y$ is an open embedding of smooth affine varieties, M is a D-module on U, and N a D-module on Y, we have

$$\mathrm{Hom}(j^!N, M) = \mathrm{Hom}_{D(U)}(O(U) \otimes_{O(Y)} N, M) =$$

$$= \mathrm{Hom}_{D(U)}(D(U) \otimes_{D(Y)} N, M) = \mathrm{Hom}_{D(Y)}(N, M) = \mathrm{Hom}(N, j_*M).$$

Thus, we get

Proposition 2.1. *For an affine open embedding j the functor $j^!$ is left adjoint to j_*.*

2.2. **The De Rham complex of a left D-module.** Recall that if M is a vector bundle on a smooth X of dimension n (C^∞, analytic, or algebraic) with a flat connection ∇, then one can define the De Rham complex $\mathrm{dR}(M)$ of X with coefficients in M:

$$0 \to M = M \otimes_{O_X} \Omega_X^0 \to M \otimes_{O_X} \Omega_X^1 \to ... \to M \otimes_{O_X} \Omega_X^n \to 0$$

with differential locally given by $d(s \otimes \omega) = \nabla(s) \otimes \omega + s \wedge d\omega$. In the algebraic case, the same formula works more generally, when M is any quasicoherent sheaf with a flat connection, i.e., a left D-module on X.

If X is affine, the cohomology of the global sections of $\mathrm{dR}(M)$ is called the *de Rham cohomology* of the D-module M, denoted by $H^i_{dR}(X, M)$. For example, if $M = O(X)$ then $\mathrm{dR}(M)$ is the usual algebraic De Rham complex of X, so $H^i_{dR}(X, O(X)) = H^i_{dR}(X)$ is the algebraic de Rham cohomology of X. By Grothendieck's algebraic De Rham theorem, for $k = \mathbb{C}$ this space is naturally isomorphic to $H^i(X, \mathbb{C})$ by the integration pairing.

Also, for $M = D_X$ we obtain a complex $\mathrm{dR}(D_X)$ of right D-modules on X.

Exercise 2.2. Using local coordinates, show that $\mathrm{dR}(D_X)$ is exact except in degree n, where its cohomology is canonically isomorphic to Ω_X as a right D-module. Thus, $\mathrm{dR}(D_X)$ is locally a projective resolution of Ω_X.

2.3. **Direct and inverse image for a map to a point.** Let $\dim X = n$ and $\pi : X \to \mathrm{pt}$. Then π^0 is obviously exact, and $\pi^!(k) = O_X[n]$.

The situation with direct image is more interesting. Namely, we have $\pi_0(M) = M \otimes_{D(X)} O(X) = M/\mathrm{Vect}(X)M$. For example,

$$\pi_0(\Omega(X)) = \Omega(X)/\mathrm{Vect}(X)\Omega(X).$$

Recall that for a top form ω we have $\mathrm{Lie}_v\omega = di_v\omega$. Since any $n-1$-form is a linear combination of forms of the form $i_v\omega$ (check it!), we have $\pi_0(\Omega(X)) = \Omega^n(X)/d\Omega^{n-1}(X) = H^n_{dR}(X)$, the n-th algebraic De Rham cohomology of X (isomorphic to $H^n(X, \mathbb{C})$ for $k = \mathbb{C}$ by Grothendieck's theorem).

Now consider the full direct image $\pi_*(\Omega(X))$ (a graded vector space). To compute it, we need a projective resolution of $\Omega(X)$ as a $D(X)$-module. We have just constructed such a resolution, namely $\mathrm{dR}(D(X))$. Thus, $\pi_*(\Omega(X))$ is the cohomology of the complex

$$\mathrm{dR}(D(X)) \otimes_{D(X)} O(X) = \mathrm{dR}(O(X)),$$

the usual de Rham complex of X (shifted by n). Thus,

$$H^{-i}(\pi_*(\Omega(X))) = H^{n-i}_{\mathrm{dR}}(X) = H^{n-i}(X, \mathbb{C})$$

for $0 \leq i \leq n$ (the latter for $k = \mathbb{C}$), and the cohomology vanishes in all the other degrees.

2.4. **Direct and inverse image for a closed embedding and Kashiwara's theorem.** Let $i : X \to Y$ be a closed embedding of smooth varieties. In this case $D_{X\to Y} = D_Y/I_X D_Y$, where $I_X \subset O_X$ is the ideal sheaf cutting out Y inside X. Thus, the functor i_0 has a particularly nice description. Namely, let us view X as a subvariety of Y using i. We may work locally and pick a coordinate system $x_1, ..., x_n, z_1, ..., z_p$ on Y such that X is locally cut out by the equations $z_1 = ... = z_p = 0$ (i.e., $p = \dim Y - \dim X$). In this case we have $i_0(M) = \oplus_{m_1,...,m_p \geq 0} M\partial_{z_1}^{m_1}...\partial_{z_p}^{m_p}$.

Given a closed subvariety $Z \subset Y$, let us say that $M \in \mathcal{M}(Y)$ is *supported on* Z if for any $f \in O(Y)$ vanishing on Z and any local section s of M, there exists $N \geq 0$ such that $f^N s = 0$. Let $\mathcal{M}_Z(Y)$ denote the category of D-modules on Y which are supported on Z. The above shows that i_0 is an exact functor $\mathcal{M}(X) \to \mathcal{M}_X(Y)$.

Theorem 2.3. *(Kashiwara) The functor $i_0 : \mathcal{M}(X) \to \mathcal{M}_X(Y)$ is an equivalence of categories, whose inverse is $i^!$. In particular, the functor $i^!|_{\mathcal{M}_X(Y)}$ is exact (i.e., has no cohomology outside of degree zero).*

Now we see the significance of the shift in the definition of $\pi^!$. Theorem 2.3 implies that the functor i^0 has cohomology in only in degree $d = \dim X - \dim Y$, so the shift by d is precisely what's needed to place this cohomology in degree zero and make the functor $i^!$ exact.

Because of limited time, we will not prove Kashiwara's theorem in general, but will treat the simplest special case $X = \{0\}$ and $Y = \mathbb{A}^1$, to which everything reduces in some sense. Let us work with left D-modules. We have $i_0(k) = \delta$, where δ is the D-module generated by the δ-function, i.e. $\delta = D/Dx$, where $D = D(\mathbb{A}^1)$. This D-module has a basis $v_i = \partial^i$, $i \geq 0$, with $\partial v_i = v_{i+1}$, $xv_i = -iv_{i-1}$, and $xv_0 = 0$. In particular, δ is simple (check it!)

Also, $i^0\delta = H^1 i^! \delta = \delta/x\delta = 0$. However, we have nontrivial cohomology in degree 0. Namely, we have $i^!\delta = k \otimes^L_{k[x]} \delta[-1]$, so $H^0 i^! \delta = \mathrm{Tor}_1^{k[x]}(k, \delta)$. By using the resolution $k[x] \to k[x]$ of k (where the map is multiplication by x), we see that this space is the kernel of the map $x : \delta \to \delta$, which is k (spanned by v_0). Thus, $H^0 i^! \delta = k$. So we see that in our special case Kashiwara's theorem reduces to

Proposition 2.4. *Any D-module on the line supported at the origin is of the form $V \otimes \delta$, where V is a vector space.*

Proof. First note that $\mathrm{Ext}^1(\delta, \delta) = 0$. This is seen by mapping the resolution $D \to D$ of δ (where the map is right multiplication by x) to δ.

Now let M be a D-module on $\mathbb{A}^1$ supported at zero. Let $V = \mathrm{Ker} x|_M$. Since $\delta = D/Dx$, we have a natural map $\alpha : V \otimes \delta \to M$. It is easy to see that α is injective (as δ is simple), and $\mathrm{Im}\alpha$ contains $\mathrm{Ker} x|_M$.

We claim that α is an isomorphism. Indeed, otherwise we can pick a nonzero element $\bar{u} \in \mathrm{Coker}\alpha$ such that $x\bar{u} = 0$. Since $\mathrm{Ext}^1(\delta, \delta) = 0$, this element admits a lift u to M such that $xu = 0$. Hence $u \in \mathrm{Im}\alpha$, a contradiction. $\square$

Exercise 2.5. Let $i : X \to Y$ be a closed embedding of smooth varieties, and let $d = \dim X - \dim Y$ (a nonpositive number). Show that the functor Li^0 on $\mathcal{M}(Y)$ has no cohomology outside of degrees $d, ..., 0$. Show that the functor $i^d := H^d Li^0$ is left exact, and $Ri^d = Li^0[d]$; in particular, $H^j Ri^d \cong H^{j+d} Li^0$ for $0 \le j \le -d$. Thus, $i^0 = H^{-d} Ri^d$.

2.5. **D-modules on singular varieties.** Kashiwara's theorem motivates a definition of the category of D-modules on a singular variety. Namely, if X is an affine variety (not necessarily smooth), then one can still define the algebra $D(X)$ of differential operators on X using Grothendieck's definition, but this algebra is in general badly behaved (e.g., not Noetherian). Thus it is not good to define D-modules on X as $D(X)$-modules. Rather, if $i : X \to Y$ is a closed embedding into a smooth variety, one should define the category $\mathcal{M}(X)$ of D-modules on X as the category $\mathcal{M}_X(Y)$ of D-modules on Y supported on X. One then has to show that this definition actually depends only on X and not on Y or i; we will skip the proof of this fact.

Since this definition is local, it extends to arbitrary varieties (not necessarily affine). Thus, for any variety X we have now defined the category $\mathcal{M}(X)$ of D-modules on X. In fact, this category is defined even more generally, when X is a scheme of finite type over k, namely $\mathcal{M}(X) = \mathcal{M}(X_{\mathrm{red}})$ (i.e. this category depends only on the reduced part of X). This definition is natural since for a closed subscheme $X \subset Y$ of a smooth variety Y, the category $\mathcal{M}_X(Y)$ by definition depends only on X_{red}. Note also that we no longer make a distinction between left and right D-modules (in fact, these notions are not even meaningful when X is singular).

2.6. **General direct images.** If $\pi : X \to Y$ is a general morphism (not necessarily affine) then the direct image functor cannot be defined as the derived functor of a right exact functor in a way compatible with compositions. This is demonstrated by the following example.

Example 2.6. Let $\pi : SL_2 \to \mathbb{P}^1 = SL_2/B$ be the natural map (where B is the subgroup of upper triangular matrices), and $\psi : \mathbb{P}^1 \to \mathrm{pt}$. Then the maps π and $\psi \circ \pi$ are affine (even though ψ is not affine). Moreover, $\pi_0(\Omega_{SL_2}) = 0$, since it computes the top cohomology of the fibers of π, i.e., $H^2(B)$, which is zero. On the other hand, $(\psi \circ \pi)_0(\Omega_{SL_2}) = H^3(SL_2) = k$. Thus, there is no functor ψ_0 such that $(\psi \circ \pi)_0 = \psi_0 \circ \pi_0$.

Let us now define the correct functor π_*. Since the problem is local in Y, let us assume first that Y is affine. We may also assume that X and Y are smooth (by embedding them into affine spaces). Let $D_{X\to Y} = \pi^0(D(Y))$. Now, given a right D-module M on X, we can define $M \otimes^L_{D_X} D_{X\to Y}$, an object in the bounded derived category of sheaves on X with a commuting action of $D(Y)$. Let us represent this object by an explicit complex. To this end, recall that $\mathrm{dR}(D_X)$ is a resolution of Ω_X. Thus, the complex $M \otimes_{O_X} \mathrm{dR}(D_X) \otimes_{O_X} \Omega_X^{-1}$ of right D_X-modules (called the Koszul complex of M) is a resolution of M by objects adapted to the functor $\otimes_{D_X} D_{X\to Y}$. Hence, the object $M \otimes^L_{D_X} D_{X\to Y}$ is represented by the complex

(4)

$$M \otimes_{O_X} \mathrm{dR}(D_X) \otimes_{O_X} \Omega_X^{-1} \otimes_{D_X} D_{X\to Y} = M \otimes_{O_X} \Omega_X^\bullet \otimes_{O_X} \Omega_X^{-1} \otimes_{O(Y)} D(Y),$$

which can also be written as $M \otimes_{O_X} \wedge^{n-\bullet} TX \otimes_{O(Y)} D(Y)$.

Note that complex (4) consists of quasicoherent sheaves (even though the differential is not O-linear). Hence, its terms have no higher sheaf cohomology if X is affine. Thus, if X were affine, we could simply take the global sections of (4) to get the object $\pi_*(M)$ in the bounded derived category of $D(Y)$-modules, which is equivalent to what we did above. However, if X is not affine, we have seen that this is not the right thing to do, since the terms of (4) may have higher sheaf cohomology. Rather, the correct definition is as follows.

Definition 2.7. The direct image $\pi_*(M)$ is the hypercohomology of $M \otimes^L_{D_X} D_{X\to Y}$:

$$\pi_*(M) := R\Gamma(M \otimes^L_{D_X} D_{X\to Y}).$$

In practice $\pi_*(M)$ may be computed by computing the hypercohomology of the complex (4) using a Cech covering $X = \cup_{i=1}^N U_i$ of X by affine open sets. Note that this definition can be applied more generally to an object in the derived category. Moreover, it can be shown that this definition is compatible with gluing on Y, which allows one to extend it to arbitrary (not necessarily affine) Y; in this case $R\Gamma$ should be replaced $R\pi_\bullet$, where $\pi_\bullet$ is the sheaf-theoretic direct image.

Altogether, we obtain an exact functor $\pi_* : D^b(\mathcal{M}(X)) \to D^b(\mathcal{M}(Y))$ for any morphism $\pi : X \to Y$ of algebraic varieties, given by

$$\pi_*(M) := R\pi_\bullet(M \otimes^L_{D_X} D_{X\to Y}).$$

Moreover, one can show that π_* enjoys nice properties which we stated in the case of affine morphisms:

Proposition 2.8. *(i) Direct image is compatible with compositions.*
(ii) If j is any open embedding then $j^!$ is left adjoint to j_*.

Note also that π_* is a composition of a left exact and a right exact functor, i.e., can be defined only in the derived category.

Example 2.9. Let Y be a point and $M = \Omega_X$. In this case, complex (4) is the algebraic De Rham complex $\Omega^\bullet_X$, so its hypercohomology is the algebraic de Rham cohomology $H^\bullet_{\mathrm{dR}}(X)$, which for $X = \mathbb{C}$ coincides with the usual cohomology $H^\bullet(X, \mathbb{C})$. The only peculiarity is that we have a shift by $n = \dim X$, namely $H^i\pi_*(\Omega_X) = H^{i+n}_{\mathrm{dR}}(X)$ for $-n \le i \le n$, i.e., the cohomology of X (which normally lives in degrees 0 to $2n$) is placed symmetrically between degrees $-n$ and n. In particular, this implies the well known fact that for smooth affine X over $\mathbb{C}$ we have $H^i(X, \mathbb{C}) = 0$ if $i > n$.

E.g., for $X = \mathbb{P}^1$, $\pi_*(\Omega_X)$ has 1-dimensional cohomology in degrees -1 and 1 and zero everywhere else.

Example 2.10. Let X be any variety (possibly singular). One can show that there exists a unique irreducible D-module IC_X on X whose restriction (=inverse image) to the smooth locus X_{sm} of X is $\Omega_{X_{\mathrm{sm}}}$. Let $\pi : X \to \mathrm{pt}$. Then $\pi_*(IC_X)$ is called the *intersection cohomology* of X, denoted $IH^\bullet(X)$. If $k = \mathbb{C}$, it coincides with the topological intersection cohomology of X introduced by Goresky and Macpherson. For this reason, the D-module IC_X is called the *intersection cohomology D-module* of X.

3. Lecture 3

3.1. **Base change.** Let $\pi : X \to Y$ be a morphism, and $\tau : S \to Y$ a base change map. Let $W = X \times_Y S$, and $\widetilde{\pi} : W \to S$, $\widetilde{\tau} : W \to X$ the corresponding natural maps.

Proposition 3.1. *We have a natural isomorphism of functors $\tau^! \circ \pi_* \cong \widetilde{\pi}_* \circ \widetilde{\tau}^!$.*

We will skip the proof.

Example 3.2. Let S be a point, and $\pi : X \to Y$ be a family of smooth varieties parametrized by Y. In this case, Proposition 3.1 implies that the cohomology of $\pi_*(\Omega_X)$ is a graded vector bundle on Y whose fiber at a point $y \in Y$ is the cohomology of the fiber $\pi^{-1}(y)$. This bundle, being a D-module on Y, is equipped with a flat connection, called the *Gauss-Manin connection.* This connection is the main object of study in Hodge theory. For example, if $Y = \mathbb{C} \setminus \{0, 1\}$, X is the surface $y^2 = x(x-1)(x-z)$ in $\mathbb{C}^3$, and $\pi : X \to Y$ is given by $\pi(x, y, z) = z$ (the "universal elliptic curve"), then the middle cohomology of $\pi_*(\Omega_X)$ is the rank 2 flat bundle on Y arising from the Picard-Fuchs differential equation (a special case of the Gauss hypergeometric equation).

3.2. **Adjunction for proper maps.** Similarly to open embeddings, for proper morphisms we also have an adjunction relation between the functors $\pi^!$ and π_*, but it is on the other side.

Proposition 3.3. *If $\pi : X \to Y$ is proper (for example, projective), then the functor $\pi^!$ is right adjoint to π_*.*

We will not prove this proposition, but will treat the special case when Y is a point, X is smooth, and M is a vector bundle with a flat connection. For this purpose, note that

$$\mathrm{Ext}^i_{D_X}(M, O_X) = \mathrm{Ext}^i_{D_X}(O_X, M^*) = H^i_{\mathrm{dR}}(X, M^*),$$

the De Rham cohomology of M^*. To see this, recall that a resolution of O_X is $\mathrm{dR}(D_X) \otimes \Omega_X^{-1}$. Taking Hom from this complex to M^*, we obtain the de Rham complex of M^*, and the Ext group in question is the hypercohomology of this complex, i.e. $H^i_{\mathrm{dR}}(M^*)$, as desired. Thus, we have $H^i\mathrm{RHom}(M, \pi^! k) = H^{i+n}_{\mathrm{dR}}(X, M^*)$, where $n = \dim X$. On the other hand, $H^i\mathrm{RHom}(\pi_*(M), k) = H^{n-i}_{\mathrm{dR}}(X, M)^*$. In general these are different things, but if X is proper then it satisfies Poincare duality, so these two things are the same: $H^{i+n}_{\mathrm{dR}}(X, M^*) \cong H^{n-i}_{\mathrm{dR}}(X, M)^*$. Thus for proper X we have the adjunction relation $\mathrm{Hom}(M, \pi^! k) = \mathrm{Hom}(\pi_* M, k)$.

3.3. **The exact triangle attached to a closed embedding.** Let $i : Z \to X$ be a closed embedding, and $j : X \backslash Z \to X$ the corresponding open embedding.

Lemma 3.4. *For any D-module (or complex of D-modules) N on $X \backslash Z$ we have $i^! j_* N = 0$.*

Proof. Indeed, using the adjunctions, $\mathrm{Hom}(M, i^! j_* N) = \mathrm{Hom}(i_* M, j_* N) = \mathrm{Hom}(j^! i_* M, N) = 0$ since $j^! i_* M = 0$. $\square$

Observe now that for any D-module M on X (or, more generally, object of $D^b(\mathcal{M}(X))$) we have the adjunction morphism $a : M \to j_*j^!M$. On $X \setminus Z$, this morphism is an isomorphism, since $j^!j_*N = N$ for any N. Thus, $\mathrm{Cone}(a)$ is supported on Z.

Proposition 3.5. *We have a natural isomorphism $\zeta : \mathrm{Cone}(a) \to i_*i^!M[1]$. In other words, we have an exact triangle*

$$i_*i^!M \to M \to j_*j^!M \tag{5}$$

in $D^b(\mathcal{M}(X))$.

Proof. Since both objects are supported on Z, by Kashiwara's theorem it suffices to construct a morphism $\widetilde{\zeta} : i^!\mathrm{Cone}(a) \to i^!i_*i^!M[1]$ and show that it is an isomorphism; then $\zeta = i_*(\widetilde{\zeta})$ is also an isomorphism. But this is easy since $i^!i_*i^!M = i^!M$ and $i^!j_*N = 0$ for any N; namely, we may take $\widetilde{\zeta} = \mathrm{Id}$. $\square$

Example 3.6. 1. Suppose M is supported on Z. Then $j^!M = 0$, while $i_*i^!M \to M$ is an isomorphism by Kashiwara's theorem.

2. Suppose X is smooth, Z is a smooth divisor in X, and M is O-coherent. Then we have a short exact sequence of D-modules on X:

$$0 \to M \to j_*j^!M \to i_*i^0M \to 0,$$

which coincides with the above exact triangle, since $i^0M = i^!M[1]$.

3.4. **Verdier duality.** Verdier duality extends the notion of the dual of a vector bundle with a flat connection to general D-modules. Namely, for smooth X of dimension n define the functor $\mathbb{D} : D^b(\mathcal{M}_l(X)) \to D^b(\mathcal{M}_l(X))$ by $\mathbb{D}(M) = \mathrm{RHom}_{D_X}(M, D_X \otimes \Omega_X^{-1})[n]$.

Let us say that M is a *coherent* D-module if it is locally finitely generated over D_X.

Proposition 3.7. *(i) $\mathbb{D}$ preserves the derived category of coherent D-modules, and on this category $\mathbb{D}^2 = \mathrm{Id}$.*

(ii) If M is a vector bundle with a flat connection then $\mathbb{D}(M) = M^*$ concentrated in degree zero.

We can also define the functor $\boxtimes$ of external tensor product (on all D-modules): if M is a D-module on X and N a D-module on Y then $M \boxtimes N$ is a D-module on $X \times Y$.

Exercise 3.8. Let X be a smooth variety and $\Delta : X \to X \times X$ be the diagonal map. Show that for coherent D-modules M, N on X (or, more generally, objects of the derived category of such D-modules) one has a natural isomorphism $\mathrm{Hom}(M, \mathbb{D}N) \cong \mathrm{Hom}(M \boxtimes N, \Delta_*O_X)$. Deduce that $\mathrm{Hom}(M, \mathbb{D}N) \cong \mathrm{Hom}(N, \mathbb{D}M)$.

3.5. **Singular support.** Assume that X is smooth affine, and let M be a coherent D-module on X. Then M has a good filtration, i.e. such that $\mathrm{gr}\, M$ is finitely generated over $\mathrm{gr}\, D(X) = O(T^*X)$ (check it!). Then we can define the support $\mathrm{Supp}(\mathrm{gr}\, M) \subset T^*X$, a closed subvariety.

Exercise 3.9. The variety $\mathrm{Supp}(\mathrm{gr}\, M)$ is independent on the choice of a good filtration on M.

Definition 3.10. The variety $\mathrm{Supp}(\mathrm{gr}\, M)$ is called the *singular support* of M and denoted $SS(M)$.

This notion can be extended to non-affine X in an obvious way.

Exercise 3.11. 1. The singular support of $O(X)$ is $X \subset T^*X$.

2. The singular support of δ is the line $p = 0$ in the x, p-plane.

3. If $M = M(x^s) = D(\mathbb{C})/D(\mathbb{C})(x\partial - s)$, $s \notin \mathbb{Z}$, then $SS(M) \subset \mathbb{C}^2$ is defined by the equation $xp = 0$. On the other hand, M is irreducible. This shows that an irreducible D-module may have reducible singular support.

4. If $M = D_X$ then $SS(M) = T^*X$.

Recall that T^*X has a natural symplectic structure. We say that a closed subvariety $Z \subset T^*X$ is *coisotropic* if for any $z \in Z$, T_zZ is a coisotropic subspace of $T_z(T^*X)$ under the symplectic form. It is clear that any component of a coisotropic subvariety has dimension $\geq \dim X$.

Theorem 3.12. *(Gabber) $SS(M)$ is coisotropic. In particular its components have dimension $\geq \dim X$.*

This theorem is rather tricky to prove, and we will skip the proof.

3.6. **Holonomic D-modules.** A coisotropic subvariety $Z \subset T^*X$ is called *Lagrangian* if all its components have the minimal possible dimension, $\dim X$. This is equivalent to saying that Z is of pure dimension $\dim X$ and the symplectic form vanishes on Z.

Definition 3.13. A coherent D-module M is called *holonomic* if $SS(M)$ is Lagrangian.

The term "holonomic" comes from the fact that if a function $f(z_1, ..., z_n)$ satisfies a holonomic system of linear differential equations with rational coefficients then it generates a holonomic D-module.

Example 3.14. The D-modules $O(X)$, $M(x^s)$, $M(e^x)$ are holonomic, while D_X is not holonomic (for $\dim X > 0$).

Proposition 3.15. *A holonomic D-module has finite length, and* $\mathrm{RHom}(M, N)$ *is finite dimensional for holonomic* M, N.

Proposition 3.16. *If* M *is holonomic then* $\mathbb{D}(M)$ *is holonomic and concentrated in degree zero.*

Exercise 3.17. 1. Let A be a finitely generated algebra with generators $a_1, ..., a_s$ and let M be a finitely generated A-module with generators $m_1, ..., m_r$. Let d_N be the dimension of the linear span of am_j, where a is a monomial in a_i of degree $\leq N$. Let $d = \limsup_{N\to\infty} \frac{\log d_N}{\log N}$. Show that d is independent on the choice of m_j and a_i. It is called the *Gelfand-Kirillov dimension* of M.

2. Show that if X is smooth affine then the algebra $D(X)$ is finitely generated.

3. Show that if M is a coherent D-module on a smooth affine X then the Gelfand-Kirillov dimension d of M equals the dimension of $SS(M)$. Deduce *Bernstein's inequality*: if $M \neq 0$ then $d \geq \dim X$, with equality achieved exactly for holonomic modules.

3.7. **Formalism of six functors.** Let $\mathcal{M}_{\mathrm{hol}}(X)$ be the category of holonomic D-modules on X. One can show that this category is well defined even when X is singular. Moreover, we have

Theorem 3.18. *The functors* π_* *and* $\pi^!$ *preserve the bounded derived category of holonomic D-modules,* $D^b(\mathcal{M}_{\mathrm{hol}}(X))$.

This allows us to introduce two more functors: $\pi_! = \mathbb{D}\pi_*\mathbb{D}$ and $\pi^* = \mathbb{D}\pi^!\mathbb{D}$ acting on $D^b(\mathcal{M}_{\mathrm{hol}}(X))$.

Exercise 3.19. Show that if X is a smooth variety over $\mathbb{C}$ and $\pi : X \to \mathrm{pt}$ then $\pi_!(\Omega_X)$ is the cohomology of X with compact supports.

The six functors $\pi_*, \pi^*, \pi_!, \pi^!, \mathbb{D}, \boxtimes$ enjoy many nice properties, some of which are summarized in the appendix. These properties are called *Grothendieck's formalism of six functors*. However, a detailed discussion of this formalism is beyond the scope of these lectures.

3.8. **The intermediate extension and classification of irreducible holonomic D-modules.** Let $j : U \to X$ be an open embedding. Then j_* is left exact (since it is right adjoint), thus $j_!$ is right exact on the category of holonomic D-modules. Moreover, we have a canonical functorial morphism $\alpha_M : j_!M \to j_*M$, since $\mathrm{Hom}(j_!M, j_*M) \cong \mathrm{Hom}(j^!j_!M, M) = \mathrm{Hom}(M, M)$, which contains Id_M. This map is nontrivial only in degree zero, so for any holonomic D-module M on U, $\mathrm{Im}\alpha_M$ is a D-module on X. It is called the *intermediate extension* of M and denoted $j_{!*}(M)$. Note that $M \mapsto j_{!*}(M)$ is a functor, but it is only additive (not left or right exact), see below.

Exercise 3.20. 1. Show that if N is a holonomic D-module on X supported on $X \setminus U$ then $\mathrm{Hom}(j_!M, N) = \mathrm{Hom}(N, j_*M) = 0$. Deduce that $j_{!*}(M)$ has no nonzero submodules or quotient modules supported on $X \setminus U$. Show that $j_{!*}(M)$ is characterized by this property (i.e., it is the unique holonomic D-module on X with this property whose restriction to U is M).

2. Show that if M is simple then so is $j_{!*}(M)$, and vice versa. In particular, if U is smooth then $j_{!*}O_U = IC_X$.

3. Show that if U is smooth then every irreducible holonomic D-module on X which is O-coherent on U is of the form $j_{!*}(M)$ for a unique irreducible O-coherent D-module M on U (vector bundle with an irreducible flat connection).

4. Deduce that any irreducible holonomic D-module N on X is of the form $i_*j_{!*}(M)$, where $i : Z \to X$ is a closed embedding, $j : U \to Z$ is an open embedding of some smooth open set U, and M is an O-coherent irreducible D-module on U.

5. Let $M = M_U(\log x)$ on $U = \mathbb{C}^*$ (a nontrivial extension of O_U by O_U), and let $X = \mathbb{C}$. Then $j_{!*}(M) = M_X(\log x)$, which has composition factors O_X, δ, O_X (in this order). Deduce that $j_{!*}$ is not left or right exact (in fact, it is not exact in the middle), and $j_{!*}(M)$ may contain nonzero subquotients supported on $X \setminus U$.

3.9. **Beilinson-Bernstein localization theorem.** In conclusion let us give an application of D-modules to representation theory.

Let $\mathfrak{g}$ be a semisimple Lie algebra over k, and X be the corresponding flag variety. We have a natural action map $\alpha : \mathfrak{g} \to \mathrm{Vect}(X)$ which induces an algebra homomorphism $\alpha : U(\mathfrak{g}) \to D(X)$, where $D(X) = \Gamma(X, D_X)$. Moreover, let I_0 be the annihilator in the center $Z(\mathfrak{g})$ of $U(\mathfrak{g})$ of the trivial representation of $\mathfrak{g}$. Then it is not hard to show that $\alpha|_{I_0} = 0$. Let $U_0(\mathfrak{g}) = U(\mathfrak{g})/I_0U(\mathfrak{g})$.

Theorem 3.21. *(Beilinson-Bernstein localization theorem)*

(i) The map $\alpha : U_0(\mathfrak{g}) \to D(X)$ is a filtered isomorphism.

(ii) The functor of global sections defines an equivalence $\mathcal{M}(X) \to U_0(\mathfrak{g}) - \mathrm{mod}$ between the category of D-modules on X and the category of modules over $U_0(\mathfrak{g})$.

This theorem plays a fundamental role in the representation theory of semisimple Lie algebras.

4. Appendix: formalism of six functors (a fact sheet)

4.1. **Functors on all (coherent) D-modules.** Let $\pi : X \to Y$ be a morphism of irreducible algebraic varieties, and $d = \dim X - \dim Y$.

• Functors defined on the derived category of all D-modules: $\pi_*, \pi^!$, $\boxtimes$. The functor $\pi^!$ is $\pi^\bullet[d]$, where $\pi^\bullet$ is the inverse image of quasicoherent sheaves, i.e. it is obtained by introducing a flat connection on the sheaf-theoretic pullback (in the case of smooth varieties).

• The functors π_* and $\pi^!$ are compatible with compositions.

• The functors π_* and $\pi^!$ are compatible with base change. That is, if $\tau : S \to Y$, $W = X \times_Y S$, $\widetilde{\pi} : W \to S$ the lift of π and $\widetilde{\tau} : W \to X$ the lift of τ then $\tau^! \circ \pi_* = \widetilde{\pi}_* \circ \widetilde{\tau}^!$.

• The functor $\mathbb{D}$ is defined on the derived category of coherent D-modules, and maps this category to its opposite. Moreover, $\mathbb{D}^2 = \mathrm{Id}$ (so $\mathbb{D}$ is an antiequivalence).

• The functor $\pi_! := \mathbb{D}\pi_*\mathbb{D}$ is defined on the coherent M such that $\pi_*\mathbb{D}M$ is coherent.

• The functor $\pi^* := \mathbb{D}\pi^!\mathbb{D}$ is defined on the coherent M such that $\pi^!\mathbb{D}M$ is coherent.

• If π is proper then π_* preserves the derived category of coherent D-modules, and on this category $\pi_!$ is defined and equals π_*. Also in this case $\pi_* = \pi_!$ is left adjoint to $\pi^!$ on the derived categories of coherent D-modules.

• If π is smooth then $\pi^!$ preserves the derived category of coherent D-modules, and on this category π^* is defined and equals $\pi^![-2d]$ (so for an étale map, in particular open embedding, $\pi^* = \pi^!$). Also, in this case π_* is right adjoint to $\pi^* = \pi^![-2d]$ on the derived categories of coherent D-modules. Finally, $\pi^![-d] = \pi^*[d]$ preserves the abelian category of coherent (and all) D-modules, and is exact.

• if π is a closed embedding then π_* is the derived functor of an exact functor on the abelian category, and preserves coherent D-modules.

• if π is affine then π_* is the derived functor of a right exact functor on the abelian category of D-modules.

• The functor $\boxtimes$ is compatible with the other functors in an obvious way.

4.2. **Functors on holonomic D-modules.** All six functors above are defined on the derived categories of holonomic D-modules. Moreover:

• $\mathbb{D}$ is an exact functor from the abelian category of holonomic D-modules to its opposite.

• $\pi_!$ is left adjoint to $\pi^!$ and π_* is right adjoint to π^*.

• if π is an open embedding then π_* is the sheaf-theoretic direct image, and it is the derived functor of a left exact functor. Similarly, $\pi_!$ is the derived functor of a right exact functor.

References

R. Hotta, K. Takeuchi, T. Tanisaki, D-modules, perverse sheaves, and representation theory, Birkhäuser, 2008.

Department of Mathematics, Massachusetts Institute of Technology, Cambridge, MA 02139, USA
E-mail address: etingof@math.mit.edu