1. Let \( j : \mathbb{A}^1 \rightarrow \mathbb{P}^1 \) be the natural embedding. Let \( p(x) \) be any non-constant polynomial and let \( M(e^p) \) denote the \( D_{\mathbb{A}^1} \)-module generated by the function \( e^p \).

   a) Prove that \( D(\mathbb{C}) \cong M(e^{-p}) \).

   b) Show that the natural map \( j_!(M(e^p)) \rightarrow j_*(M(e^p)) \) is an isomorphism (hint: prove that \( j_*(M(e^p)) \) is irreducible and then use (a)).

2. Let \( j : \mathbb{G}_m \rightarrow \mathbb{A}^1 \). Show that the direct image of \( j_!(\mathcal{O}) \) to the point is zero.

3. Let \( M \) denote the following \( D \)-module on \( \mathbb{G}_m \): it consists of expressions \( p(x, \lambda)x^{i} + \lambda \) where \( p \in k[x, x^{-1}, \lambda] \) (note that we do not allow to divide by \( \lambda \)). Define \( \mathcal{E}_n = M/\lambda^nM \) (again considered as a \( D \)-module on \( \mathbb{G}_m \)). Note that multiplication by \( \lambda \) induces a nilpotent endomorphism of \( \mathcal{E}_n \).

   a) Show that \( \mathcal{E}_n \) is \( \mathcal{O} \)-coherent of rank \( n \) and that every irreducible subquotient of \( \mathcal{E}_n \) is isomorphic to \( \mathcal{O} \) (i.e. that \( \mathcal{E}_n \) is a successive extension of \( n \) copies on \( \mathcal{O} \)).

   b) Show that \( \mathcal{E}_n \) is indecomposable.

   c) Show that \( \mathcal{E}_n \) is uniquely determined by the conditions a and b (up to an isomorphism).

   d) Explain the existence and uniqueness of \( \mathcal{E}_n \) ”topologically” (using the notion of monodromy).

4. In this problem we want to compute \( j_!(\mathcal{E}_n) \). Here \( j \) is the embedding of \( \mathbb{G}_m \) into \( \mathbb{A}^1 \).

   a) Show that there exists an indecomposable \( D_{\mathbb{A}^1} \)-module \( N \) satisfying the following conditions: there exist short exact sequences

   \[
   0 \rightarrow j_!(\mathcal{O}_\mathbb{G}_m) \rightarrow N \rightarrow \mathcal{O}_{\mathbb{A}^1} \rightarrow 0
   \]

   and

   \[
   0 \rightarrow \mathcal{O}_{\mathbb{A}^1} \rightarrow N \rightarrow j_!\mathcal{O}_{\mathbb{G}_m} \rightarrow 0.
   \]

   In particular, \( N \) has \( \mathcal{O}_{\mathbb{A}^1} \) as both submodule and a quotient module and \( \delta_0 \) as a subquotient (sitting ”between” the two \( \mathcal{O} \)'s). Construct \( N \) both explicitly and by computing the correspondin Ext-groups.

   b) Prove that \( \delta \) is neither a submodule, nor a quotient of \( N \).

   c) Show that a) and b) imply that \( N = j_!(\mathcal{E}_2) \). This example shows that in general when \( j : U \rightarrow X \) is an open embedding the module \( j_!(M) \) may have subquotients concentrated on \( X \setminus U \) (we only know that it has neither quotients nor submodules concentrated on the complement).

   d) Explain how \( j_!(\mathcal{E}_n) \) looks like.

5. Let \( a_1, \ldots, a_n \) be generic complex numbers, and \( X \) be the open set in \( \mathbb{C}^{n+1} \) with coordinates \( t, z_1, \ldots, z_n \), defined by the inequalities \( t \neq z_i, z_i \neq z_j \). Let \( Y \) be the open set in \( \mathbb{C}^n \) defined by the inequalities \( z_i \neq z_j \), and let \( \pi : X \rightarrow Y \) be the map sending

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1 Problems 1, 2, 3, 4 were proposed by A. Braverman.
(t, z_1, ..., z_n) to (z_1, ..., z_n). Let L be the $\mathcal{O}$-coherent D-module on X generated by the function $\psi = \prod_{i=1}^n (t - z_i)^{a_i}$. 

(a) Compute $\pi^\ast(L)$. Namely, show that $\pi^\ast(L)$ is an $\mathcal{O}$-coherent D-module on Y of rank $n - 1$, which is a trivial vector bundle on Y, and calculate the corresponding Gauss-Manin connection on this bundle. (You will obtain the simplest nontrivial case of the so called Knizhnik-Zamolodchikov equations).

(b) Provide integral formulas for flat sections of $\pi^\ast(L)$ (using Pochhammer loops).