18.769: Algebraic D-modules. Fall 2013

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Problem set 6 (due Tuesday, December 10) [1]

1. Let $j : \mathbb{A}^1 \to \mathbb{P}^1$ be the natural embedding. Let $p(x)$ be any non-constant polynomial and let $M(e^p)$ denote the $\mathcal{D}_{\mathbb{A}^1}$-module generated by the function e^p.

a) Prove that $\mathbb{D}(M(e^p)) \simeq M(e^{-p})$.

b) Show that the natural map $j_!(M(e^p)) \to j_*(M(e^p))$ is an isomorphism (hint: prove that $j_*(M(e^p))$ is irreducible and then use (a)).

2. Let $j : \mathbb{G}_m \to \mathbb{A}^1$. Show that the direct image of $j_!(\mathcal{O})$ to the point is zero.

3. Let M denote the following $\mathcal{D}$-module on $\mathbb{G}_m$: it consists of expressions $p(x,\lambda)x^{\lambda+i}$ where $p \in k[x, x^{-1}, \lambda]$ (note that we do not allow to divide by λ). Define $\mathcal{E}_n = M/\lambda^n M$ (again considered as a $\mathcal{D}$-module on $\mathbb{G}_m$). Note that multiplication by λ induces a nilpotent endomorphism of $\mathcal{E}_n$.

a) Show that $\mathcal{E}_n$ is $\mathcal{O}$-coherent of rank n and that every irreducible subquotient of $\mathcal{E}_n$ is isomorphic to $\mathcal{O}$ (i.e. that $\mathcal{E}_n$ is a successive extension of n copies on $\mathcal{O}$).

b) Show that $\mathcal{E}_n$ is indecomposable.

c) Show that $\mathcal{E}_n$ is uniquely determined by the conditions a and b (up to an isomorphism).

d) Explain the existence and uniqueness of $\mathcal{E}_n$ "topologically" (using the notion of monodromy).

4. In this problem we want to compute $j_{!*}(\mathcal{E}_n)$. Here j is the embedding of $\mathbb{G}_m$ into $\mathbb{A}^1$.

a) Show that there exists an indecomposable $\mathcal{D}_{\mathbb{A}^1}$-module N satisfying the following conditions: there exist short exact sequences

$$0 \to j_*\mathcal{O}_{\mathbb{G}_m} \to N \to \mathcal{O}_{\mathbb{A}^1} \to 0$$

and

$$0 \to \mathcal{O}_{\mathbb{A}^1} \to N \to j_!\mathcal{O}_{\mathbb{G}_m} \to 0.$$

In particular, N has $\mathcal{O}_{\mathbb{A}^1}$ as both submodule and a quotient module and δ_0 as a subquotient (sitting "between" the two $\mathcal{O}$'s). Construct N both explicitly and by computing the correspondin Ext-groups.

b) Prove that δ is neither a submodule, nor a quotient of N.

c) Show that a) and b) imply that $N = j_{!*}(\mathcal{E}_2)$. This example shows that in general when $j : U \to X$ is an open embedding the module $j_{!*}(M)$ may have subquotients concentrated on $X \backslash U$ (we only know that it has neither quotients nor submodules concentrated on the complement).

d) Explain how $j_{!*}(\mathcal{E}_n)$ looks like.

5. Let $a_1, ..., a_n$ be generic complex numbers, and X be the open set in $\mathbb{C}^{n+1}$ with coordinates $t, z_1, ..., z_n$, defined by the inequalities $t \neq z_i$, $z_i \neq z_j$. Let Y be the open set in $\mathbb{C}^n$ defined by the inequalities $z_i \neq z_j$, and let $\pi : X \to Y$ be the map sending

[1] Problems 1, 2, 3, 4 were proposed by A. Braverman.

$(t, z_1, ..., z_n)$ to $(z_1, ..., z_n)$. Let L be the $\mathcal{O}$-coherent D-module on X generated by the function $\psi = \prod_{i=1}^n (t - z_i)^{a_i}$.

(a) Compute $\pi_*(L)$. Namely, show that $\pi_*(L)$ is an $\mathcal{O}$-coherent D-module on Y of rank $n - 1$, which is a trivial vector bundle on Y, and calculate the corresponding Gauss-Manin connection on this bundle. (You will obtain the simplest nontrivial case of the so called Knizhnik-Zamolodchikov equations).

(b) Provide integral formulas for flat sections of $\pi_*(L)$ (using Pochhammer loops).