1. Let $G$ be a finite group acting faithfully on a smooth irreducible affine complex algebraic variety $X$.

(a) Show that a $G$-equivariant $D$-module on $X$ is the same thing as a module over the algebra $A := \mathbb{C}[G] \ltimes D(X)$.

(b) Prove that $A$ is a simple algebra.

(c) Let $e = \frac{1}{|G|} \sum_{g \in G} g$ be the symmetrizer. Prove that the functor $M \mapsto eM = \mathbb{C}G$ defines an equivalence from the category of $G$-equivariant $D$-modules on $X$ to the category of $D(X)^G$-modules.

2. Keep the notation of Problem 1. By Noether’s theorem, the algebra $\mathcal{O}(X)^G$ is finitely generated, so it defines an algebraic variety $X/G$ (which in general is singular). One can show that points of $X/G$ bijectively correspond to $G$-orbits on $X$, which motivates the notation.

Let us say that $g \in G$ is a reflection if the fixed point set $X^g$ has a component of codimension 1 in $X$.

(a) Show that if $G$ does not contain reflections, then the natural homomorphism $\phi : D(X)^G \rightarrow D(X/G)$ is an isomorphism (where for a variety $Y$, $D(Y)$ denotes the algebra of Grothendieck differential operators on $Y$). Deduce that in this case $D(X/G)$ is Noetherian on both sides.

(b) Is $\phi$ an isomorphism in general (i.e. if $G$ may contain reflections)?

(c) Use (a) to explicitly describe $D(Y)$ when $Y$ is the quadratic cone $xy = z^2$ in the 3-dimensional space.

(d) In (c), is the functor $\Gamma$ of global sections an equivalence from the category of right $D$-modules on $Y$ to the category of right $D(Y)$-modules?

Hint: consider the modules concentrated at the vertex of the cone in both categories.

(e) Show that for any $X, G, X/G$ is locally isomorphic to $X'/G'$, where $X'$ is smooth and $G'$ does not contain reflections.

Hint. Use Chevalley’s theorem that if $G$ is a subgroup of $GL(V)$ generated by reflections, then $V/G$ is an affine space (equivalently, is smooth).
(f) Show that for any $X, G$, the algebra $D(X/G)$ is Noetherian on both sides.

3. (a) Let $X$ be a smooth irreducible variety over the complex field. Compute $\text{Tor}_{i}^{D(X)}(\Omega(X), \mathcal{O}(X))$, where $\Omega(X)$ and $\mathcal{O}(X)$ are the right (resp. left) $D(X)$-modules of top forms and functions on $X$, respectively.

(b) Recall that the Hochschild homology of an algebra $A$ is

$$HH_i(A, A) := \text{Tor}_{i}^{A-\text{bimod}}(A, A).$$

Compute $HH_i(D(X), D(X))$ for affine $X$ (apply (a) and Kashiwara’s theorem for the diagonal embedding).

4. Let $\mathcal{A}$ be an abelian category. Assume that $D(\mathcal{A})$ is equivalent to $\mathcal{C}_0(\mathcal{A})$. Prove that $\mathcal{A}$ is semi-simple.

5. Let $\mathcal{A}$ be a full abelian subcategory of an abelian category $\mathcal{B}$. Denote by $D^b(\mathcal{B})$ consisting of all complexes in $\mathcal{B}$ whose cohomologies lie in $\mathcal{A}$. We have the obvious functor $D^b(\mathcal{A}) \rightarrow D^b(\mathcal{B})$.

a) Is this functor always an equivalence of categories?

b) Prove that if the above functor is an equivalence of categories then $\mathcal{A}$ satisfies Serre’s condition: for every short exact sequence $0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0$ in $\mathcal{B}$ such that $X, Z \in \mathcal{A}$ we have $Y \in \mathcal{A}$.

c) Show that the converse of b) is still not true in general (hint: take $\mathcal{B}$ to be the category of $g$-modules where $g$ is a semi-simple Lie algebra over $\mathbb{C}$ and take $\mathcal{A}$ to be the category of finite-dimensional modules).

d) Let $R$ be a ring. Take $\mathcal{B}=$the category of left $R$-modules, $\mathcal{A}=$the category of finitely generated $R$-modules. What can you say about this case?

6. Let $X$ be a scheme of finite type over a field $k$. Let $\mathcal{A}$ denote the category of quasi-coherent sheaves on $X$ and let $\mathcal{B}$ denote the category of all sheaves of $\mathcal{O}_X$-modules. Show that in this case the functor $D^+(\mathcal{A}) \rightarrow D^+_A(\mathcal{B})$ is an equivalence of categories. As a corollary we see that if $\mathcal{F}$ is a quasicoherent sheaf then $H^i(X, \mathcal{F})$ computed in the category of all sheaves or in the category of quasi-coherent sheaves is the same.