18.769: Algebraic D-modules. Fall 2013

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Problem set 5 (due Tuesday, November 25) [1]

1. Let G be a finite group acting faithfully on a smooth irreducible affine complex algebraic variety X.

(a) Show that a G-equivariant D-module on X is the same thing as a module over the algebra $A := \mathbb{C}[G] \ltimes D(X)$.

(b) Prove that A is a simple algebra.

Hint: Assume that I is a nonzero ideal in A, and let $z = g_0b_0 + ... + g_mb_m$ be the shortest (i.e., with smallest m) nonzero element of I ($g_i \in G$, $b_i \in D(X)$). Show that one may assume that $g_0 = 1$ and b_0 is a nonzero function on X, and the possible functions b_0 together with 0 form an ideal in $\mathcal{O}(X)$ invariant under $\mathrm{Vect}(X)$. Deduce that one may assume that $b_0 = 1$. Then consider the commutator of z with a vector field to lower m if $m > 0$. Deduce that $m = 0$, and $I = A$.

(c) Let $e = \frac{1}{|G|}\sum_{g\in G} g$ be the symmetrizer. Prove that the functor $M \mapsto eM = M^G$ defines an equivalence from the category of G-equivariant D-modules on X to the category of $D(X)^G$-modules.

2. Keep the notation of Problem 1. By Noether's theorem, the algebra $\mathcal{O}(X)^G$ is finitely generated, so it defines an algebraic variety X/G (which in general is singular). One can show that points of X/G bijectively correspond to G-orbits on X, which motivates the notation.

Let us say that $g \in G$ is a reflection if the fixed point set X^g has a component of codimension 1 in X.

(a) Show that if G does not contain reflections, then the natural homomorphism $\phi : D(X)^G \to D(X/G)$ is an isomorphism (where for a variety Y, $D(Y)$ denotes the algebra of Grothendieck differential operators on Y). Deduce that in this case $D(X/G)$ is Noetherian on both sides.

Hint: use that if $U \subset X$ is an open set with $X \setminus U$ of codimension ≥ 2 then a section over U of any vector bundle on X uniquely extends to all of X).

(b) Is ϕ an isomorphism in general (i.e. if G may contain refklections)?

(c) Use (a) to explicitly describe $D(Y)$ when Y is the quadratic cone $xy = z^2$ in the 3-dimensional space.

(d) In (c), is the functor Γ of global sections an equivalence from the category of right D-modules on Y to the category of right $D(Y)$-modules?

Hint: consider the modules concentrated at the vertex of the cone in both categories.

(e) Show that for any X, G, X/G is locally isomorphic to X'/G', where X' is smooth and G' does not contain reflections.

Hint. Use Chevalley's theorem that if G is a subgroup of $GL(V)$ generated by reflections, then V/G is an affine space (equivalently, is smooth).

[1] Problems 4, 5, 6 were proposed by A. Braverman.

(f) Show that for any X, G, the algebra $D(X/G)$ is Noetherian on both sides.

3. (a) Let X be a smooth irreducible variety over the complex field. Compute $\mathrm{Tor}_i^{D(X)}(\Omega(X), \mathcal{O}(X))$, where $\Omega(X)$ and $\mathcal{O}(X)$ are the right (resp. left) $D(X)$-modules of top forms and functions on X, respectively.

(b) Recall that the Hochschild homology of an algebra A is

$$HH_i(A, A) := \mathrm{Tor}_i^{A-\mathrm{bimod}}(A, A).$$

Compute $HH_i(D(X), D(X))$ for affine X (apply (a) and Kashiwara's theorem for the diagonal embedding).

4. Let $\mathcal{A}$ be an abelian category. Assume that $D(\mathcal{A})$ is equivalent to $\mathcal{C}_0(\mathcal{A})$. Prove that $\mathcal{A}$ is semi-simple.

5. Let $\mathcal{A}$ be a full abelian subcategory of an abelian category $\mathcal{B}$. Denote by $D^b_{\mathcal{A}}(\mathcal{B})$ consisting of all complexes in $\mathcal{B}$ whose cohomologies lie in $\mathcal{A}$. We have the obvious functor $D^b(\mathcal{A}) \to D^b_{\mathcal{A}}(\mathcal{B})$.

a) Is this functor always an equivalence of categories?

b) Prove that if the above functor is an equivalence of categories then $\mathcal{A}$ satisfies Serre's condition: for every short exact sequence $0 \to X \to Y \to Z \to 0$ in $\mathcal{B}$ such that $X, Z \in \mathcal{A}$ we have $Y \in \mathcal{A}$.

c) Show that the converse of b) is still not true in general (hint: take $\mathcal{B}$ to be the category of $\mathfrak{g}$-modules where $\mathfrak{g}$ is a semi-simple Lie algebra over $\mathbb{C}$ and take $\mathcal{A}$ to be the category of finite-dimensional modules).

d) Let R be a ring. Take $\mathcal{B}$=the category of left R-modules, $\mathcal{A}$=the category of finitely generated R-modules. What can you say about this case?

6. Let X be a scheme of finite type over a field k. Let $\mathcal{A}$ denote the category of quasi-coherent sheaves on X and let $\mathcal{B}$ denote the category of all sheaves of $\mathcal{O}_X$-modules. Show that in this case the functor $D^+(\mathcal{A}) \to D^+_{\mathcal{A}}(\mathcal{B})$ is an equivalence of categories. As a corollary we see that if $\mathcal{F}$ is a quasicoherent sheaf then $H^i(X, \mathcal{F})$ computed in the category of all sheaves or in the category of quasi-coherent sheaves is the same.