18.769: Algebraic D-modules. Fall 2013

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Problem set 3 (due Thursday, October 17)[1]

1. Let X be a smooth irreducible affine variety over an algebraically closed field k of characteristic zero.

(a) Show that the algebra $D(X)$ of differential operators on X is simple.

(b) Show that $D(X)^\times = \mathcal{O}(X)^\times$, where $A^\times$ denotes the group of invertible elements of a ring A. (Hint: look at the associated graded algebra).

(c) Let $K_X = \Omega_X^n$ be the canonical line bundle of X. Show that $D(X)^{\rm op}$ is canonically isomorphic as a filtered algebra to $D(K_X)$, the algebra of differential operators from K_X to K_X.

(d) Deduce that $D(X) \cong D(X)^{\rm op}$ as a filtered algebra if and only if K_X admits a flat connection. (Use that any automorphism of $\mathcal{O}(X)$ lifts to an automorphism of $D(X)$). Deduce that if X is a curve, then $D(X) \cong D(X)^{\rm op}$ as a filtered algebra.

(e) Show that the canonical equivalence of categories between left and right $D(X)$-modules comes from an antiautomorphism of $D(X)$ if and only if X is a Calabi-Yau variety, i.e. has a nonvanishing volume form. Give an example of an affine smooth curve which does not have a nonvanishing volume form (consider a hyperelliptic curve of genus 2 with a generic missing point).

(f) Give an example of a smooth affine surface X such that K_X does not admit a flat connection, and hence $D(X)$ is not isomorphic to $D(X)^{\rm op}$ as filtered algebras. (take $Y = \mathbb{P}^1 \times \mathbb{P}^1$, embed it in $\mathbb{P}^5$ using the very ample line bundle $\mathcal{O}(1) \boxtimes \mathcal{O}(2)$, and let $X = Y \cap L^c$, where $L \subset \mathbb{P}^5$ is a generic hyperplane, and L^c is the complement of L).

(g) Show that in (f), one can ensure that X is a closed subvariety of $(\mathbb{C}^*)^n$. (Pick a generic linear coordinate system in $\mathbb{P}^5$ and redefine X to be obtained from Y by deleting the coordinate hyperplanes; show that K_X still does not have a flat connection).

(h) Use (b) to show that in the situation of (g), $D(X)$ is not isomorphic to $D(X)^{op}$ as an algebra, without regard for the filtration. (Use that an antiautomorphism of an algebra A has to map the centralizer of $A^\times$ to itself).

2. Do the exercises on p.34-35 of A. Braverman's notes
http://www.math.harvard.edu/~gaitsgde/grad_2009/Dmod_brav.pdf

3. a) Let X be given by equations $xy = 0$ in $\mathbb{A}^2$. What is $D(X)$? What is ${\rm gr}D(X)$? Is it noetherian?

b)* Do the same for the quadratic cone $xy = z^2$.

[1]Problems 2 and 3 were proposed by A. Braverman.