18.769: Algebraic D-modules. Fall 2013

Instructor: Pavel Etingof

Problem set 2 (due Thursday, October 3) [1]

1. Let $M = k[x, x^{-1}]$ ($\mathrm{char} k = 0$) with the natural structure of a $\mathcal{D}(\mathbb{A}^1)$-module. Note that we have a short exact sequence $0 \to \mathcal{O} \to M \to \delta \to 0$ where δ is the module of delta-functions at 0. Compute explicitly (e.g. by generators and relations or by writing an action of $\mathcal{D}$ on some explicit vector space) the module $\mathbb{D}(M)$ and explain how to see from this description that there is a short exact sequence $0 \to \delta \to \mathbb{D}(M) \to \mathcal{O} \to 0$.

2. Let $\widehat{\mathcal{D}}$ be the algebra of differential operators in one variable whose coefficients are formal power series (i.e. of operators of the form

$$L = a_n(x)\partial^n + ... + a_1(x)\partial + a_0(x),$$

where $a_i \in k[[x]]$). Show that for k of characteristic zero, any $\widehat{\mathcal{D}}$-module M which is finitely generated over $k[[x]]$ is of the form $k[[x]]^n$ for a unique n (with the standard componentwise action of differential operators). Is this true if you replace $k[[x]]$ with the ring of entire functions (over $k = \mathbb{C}$)? with the polynomial ring $k[x]$?

3. In this and the next problem we work over $\mathbb{C}$. Let f be a smooth function on some line interval, and $M_f := \mathcal{D}f$ (where $\mathcal{D} = \mathcal{D}(\mathbb{A}^1)$).

a) Show that M_f is holonomic if and only if f satisfies a linear differential equation with rational coefficients and coefficient 1 at the highest derivative. Moreover, it is $\mathcal{O}$-coherent outside of the poles of these coefficients.

b) Suppose M is holonomic. Is it true that if f is an entire function then M_f is $\mathcal{O}$-coherent on the whole complex plane?

c) Compute the composition factors of $M_{\log(x)}$, with multiplicities. What is the length of this $\mathcal{D}$-module? Do the same for $M_{\log(x)^n}$.

d) Do the same as in (c) for the dilogarithm function

$$L(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^2}.$$

e)* Can you compute the successive quotients of the socle filtration in (c) and (d)? (Recall that if M is a finite length module over a ring then the socle filtration on M is defined inductively by the condition that $F_iM/F_{i-1}M$ is the socle (i.e., maximal semisimple submodule) of $M/F_{i-1}M$). What is the length of this filtration?

4. Let G denote the symplectic group $Sp(2n, \mathbb{C})$. Note that G acts on $\mathcal{D} = \mathcal{D}(\mathbb{A}^n)$ by automorphisms (namely, it acts on the $2n$-dimensional vector space $\mathrm{span}(x_i, \frac{\partial}{\partial x_j})$ which generates $\mathcal{D}$ and symplectic transformations (with respect to the standard symplectic form) extend to automorphisms of $\mathcal{D}$). Thus if M is a $\mathcal{D}$-module and $g \in G$ we denote by M^g the same module but with $\mathcal{D}$-action twisted by g.

Let M be a holonomic $\mathcal{D}$-module on $\mathbb{A}^n$ and let $c = c(M)$.

[1] Problems 1 and 4 were proposed by A. Braverman.

a) Show that for generic $g \in G$ the module M^g is $\mathcal{O}$-coherent of rank c (hint: think what the transformation $M \mapsto M^g$ does to the arithmetic singular support of M and then formulate a sufficient condition for $\mathcal{O}$-coherence in terms of the arithmetic singular support).

Note that this gives another definition of c.

b) Explain what a) says explictly for $M = \delta_a$ (where $a \in \mathbb{C}$).

c) Let $\mathcal{S}(\mathbb{R}^n)$ denote the Schwartz space of $\mathbb{R}^n$, i.e. the space of functions all of whose partial derivatives are rapidly decreasing at ∞. Let $\mathcal{S}^*(\mathbb{R}^n)$ denote the appropriate topological dual space. It has a natural structure of $\mathcal{D}$-module. This is the space of tempered distributions on $\mathbb{R}^n$. The Stone-von Neumann theorem says (in particular) that $(\mathcal{S}^*(\mathbb{R}^n))^g$ is isomorphic to $\mathcal{S}^*(\mathbb{R}^n)$ for every $g \in Sp(2n, \mathbb{R})$ (the corresponding isomorphisms were explictly written by Weil). Show that for every holonomic $\mathcal{D}$-module M we have

$$\dim \operatorname{Hom}_{\mathcal{D}}(M, \mathcal{S}^*(\mathbb{R}^n)) \leq c(M).$$

In other words, the space of solutions of M in the space of tempered distributions has dimension $\leq c(M)$ (hint: use a).