18.769: Algebraic D-modules. Fall 2013
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Problem set 1 (due Thursday, September 19)

Below k is a field of characteristic 0. All affine spaces are over k. The purpose of this problem set is to gain some intuition on the relation between standard functions (or distributions) and D-modules. Problems marked with $*$ are more difficult. [1]

1. For $\lambda \in k$ let $M(x, \lambda)$ denote the $\mathcal{D}(\mathbb{A}^1)$-module with basis $x^{\lambda+i}$ for all $i \in \mathbb{Z}$ with the standard action of differential operators (note that here λ is an actual element of k and not a variable). We showed in class that $M(x, \lambda)$ is holonomic.

a) Show that $M(x, \lambda)$ is irreducible if and only if $\lambda \notin \mathbb{Z}$.

b) There is an obvious homomorphism $\phi : \mathcal{D}(\mathbb{A}^1)/\mathcal{D}(\mathbb{A}^1)(x\partial-\lambda) \to M(x, \lambda)$ sending 1 to x^λ. For which λ is it an isomorphism?

c)* Try to generalize a) to the case of an arbitrary polynomial p in n variables. (Hint: show that $M(p, \lambda) := \mathbb{C}[x_1, ..., x_n, p^{-1}]p^\lambda$ is irreducible iff λ is not an integer translate of a root of the Bernstein-Sato polynomial $b(\lambda)$ of p).

2. Let M be the D-module on $\mathbb{A}^2$ (with coordinates (x, y)) "generated by the function $e^{x/y}$", i.e. M consists of all expressions of the form $y^n pe^{x/y}$ where $n \in \mathbb{Z}$, $p \in \mathbb{C}[x, y]$ subject to the relation $y^{n+1}pe^{x/y} = y^n(y \cdot p)e^{x/y}$. The action of differential operators is standard.

a) Show that M is holonomic and irreducible.

b)* Compute the (geometric) singular support of M.

3. In this problem we consider D-modules on $\mathbb{A}^1$ with coordinate x. For every $a \in \mathbb{A}^1$ we let δ_a denote the corresponding D-module of δ-functions. In other words, δ_a has a basis $\{\delta_a^{(n)}\}_{n=0}^\infty$ and the action of differential operators is given by

$$x\delta_a^{(n)} = -n\delta_a^{(n-1)} + a\delta_a^{(n)} \quad \partial\delta_a^{(n)} = \delta_a^{(n+1)}$$

where $(x - a)\delta_a^{(0)} = 0$. We have seen that δ_a is irreducible and holonomic.

Let $\mathcal{O} = k[x]$ with the standard action of differential operators. Show that $\mathrm{Ext}^1(\delta_0, \mathcal{O})$ (as D-modules) and $\mathrm{Ext}^1(\mathcal{O}, \delta_0)$ are isomorphic to k (hint: construct explicitly the corresponding extensions $0 \to \mathcal{O} \to M \to \delta_0 \to 0$ and $0 \to \delta_0 \to M \to \mathcal{O} \to 0$; you may want to look at $\mathcal{D}(\mathbb{A}^1)/(x\partial - \lambda)\mathcal{D}(\mathbb{A}^1)$ for integer λ).

4. Let $f(x)$ be a nonzero rational function in one complex variable, and let N_f be the D-module on the affine line generated by (a branch of) the multivalued analytic function

$$\exp\left(\int f(x)dx\right).$$

a) Show that N_f is holonomic. For which f is N_f isomorphic to the quotient $\mathcal{D}(\mathbb{A}^1)/\mathcal{D}(\mathbb{A}^1)(Q\partial - P)$, where $f = P/Q$ and P, Q are relatively prime polynomials?

b) Find the composition factors of N_f and the number $c = c(N_f)$.

c) For which f, g is N_f isomorphic to N_g?

[1] Problems 1-3 were composed by A. Braverman in 2002