18.769: Algebraic D-modules. Fall 2013
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Problem set 1 (due Thursday, September 19)

Below $k$ is a field of characteristic 0. All affine spaces are over $k$. The purpose of this problem set is to gain some intuition on the relation between standard functions (or distributions) and $D$-modules. Problems marked with * are more difficult. 1

1. For $\lambda \in k$ let $M(x, \lambda)$ denote the $\mathcal{D}(\mathbb{A}^1)$-module with basis $x^{\lambda+i}$ for all $i \in \mathbb{Z}$ with the standard action of differential operators (note that here $\lambda$ is an actual element of $k$ and not a variable). We showed in class that $M(x, \lambda)$ is holonomic.
   a) Show that $M(x, \lambda)$ is irreducible if and only if $\lambda \notin \mathbb{Z}$.
   b) There is an obvious homomorphism $\phi: \mathcal{D}(\mathbb{A}^1)/\mathcal{D}(\mathbb{A}^1)(x\partial - \lambda) \to M(x, \lambda)$ sending 1 to $x^i$. For which $\lambda$ is it an isomorphism?
   c)* Try to generalize a) to the case of an arbitrary polynomial $p$ in $n$ variables. (Hint: show that $M(p, \lambda) := \mathbb{C}[x_1, ..., x_n, p^{-1}]p^\lambda$ is irreducible iff $\lambda$ is not an integer translate of a root of the Bernstein-Sato polynomial $b(\lambda)$ of $p$).

2. Let $M$ be the $D$-module on $\mathbb{A}^2$ (with coordinates $(x,y)$) ”generated by the function $e^{x/y}$, i.e. $M$ consists of all expressions of the form $y^npe^{x/y}$ where $n \in \mathbb{Z}$, $p \in \mathbb{C}[x,y]$ subject to the relation $y^{n+1}pe^{x/y} = y^n(y \cdot p)e^{x/y}$. The action of differential operators is standard.
   a) Show that $M$ is holonomic and irreducible.
   b)* Compute the (geometric) singular support of $M$.

3. In this problem we consider $D$-modules on $\mathbb{A}^1$ with coordinate $x$. For every $a \in \mathbb{A}^1$ we let $\delta_a$ denote the corresponding $D$-module of $\delta$-functions. In other words, $\delta_a$ has a basis $\{a^{(n)}\}_{n=0}^\infty$ and the action of differential operators is given by
   \[ x\delta_a^{(n)} = -n\delta_a^{(n-1)} + a\delta_a^{(n)} \quad \partial\delta_a^{(n)} = \delta_a^{(n+1)} \]
   where $(x - a)\delta_a^{(0)} = 0$. We have seen that $\delta_a$ is irreducible and holonomic.

   Let $\mathcal{O} = k[x]$ with the standard action of differential operators. Show that $\text{Ext}^1(\delta_0, \mathcal{O})$ (as $D$-modules) and $\text{Ext}^1(\mathcal{O,}\delta_0)$ are isomorphic to $k$ (hint: construct explicitly the corresponding extensions $0 \to \mathcal{O} \to M \to \delta_0 \to 0$ and $0 \to \delta_0 \to M \to \mathcal{O} \to 0$; you may want to look at $\mathcal{D}(\mathbb{A}^1)/(x\partial - \lambda)\mathcal{D}(\mathbb{A}^1)$ for integer $\lambda$).

4. Let $f(x)$ be a nonzero rational function in one complex variable, and let $N_f$ be the $D$-module on the affine line generated by (a branch of) the multivalued analytic function
   \[ \exp \left( \int f(x)dx \right). \]
   a) Show that $N_f$ is holonomic. For which $f$ is $N_f$ isomorphic to the quotient $\mathcal{D}(\mathbb{A}^1)/\mathcal{D}(\mathbb{A}^1)(Q\partial - P)$, where $f = P/Q$ and $P, Q$ are relatively prime polynomials?
   b) Find the composition factors of $N_f$ and the number $c = c(N_f)$.
   c) For which $f, g$ is $N_f$ isomorphic to $N_g$?

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1Problems 1-3 were composed by A. Braverman in 2002