Cyclotomic DAHA

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1. Cyclotomic rational Cherednik algebra (RCA) and its partially spherical subalgebra.

Definition 0.1. The cyclotomic rational Cherednik algebra for the group $S_N \ltimes (\mathbb{Z}/l\mathbb{Z})^N$, $\mathbb{H}\mathbb{H}_N^{l,\mathrm{cyc}}(c, \hbar, k)$ (where $c = (c_0, ..., c_{l-1})$) is the algebra generated by the group $S_N \ltimes (\mathbb{Z}/l\mathbb{Z})^N$, elements x_i, and the Dunkl-Opdam operators

$$D_{i,\mathrm{cyc}} = \hbar\partial_i - \frac{1}{x_i}\sum_{j=0}^{l-1} c_j\sigma_i^j - k\sum_{r\neq i,m}\frac{1}{x_i - \zeta^m x_r}(1 - s_{ir}\sigma_i^m\sigma_r^{-m}),$$

for $i = 1, ..., N$, where $\zeta = e^{2\pi i/l}$ and $\sigma_i x_j = \zeta^{\delta_{ij}} x_j$.

Let $\mathbf{p}$ be the symmetrizer of the subgroup $(\mathbb{Z}/l\mathbb{Z})^N$, and $\mathbb{H}\mathbb{H}_N^{l,\mathrm{psc}}(c, \hbar, k) = \mathbf{p}\mathbb{H}\mathbb{H}_N^{l,\mathrm{cyc}}(c, \hbar, k)\mathbf{p}$ be the corresponding partially spherical subalgebra.

One can check that the localization $\mathbf{p}\mathbb{H}\mathbb{H}_N^{l,\mathrm{cyc}}(c, \hbar, k)\mathbf{p}[\frac{1}{\prod_i x_i^\ell}]$ is isomorphic to the degenerate DAHA of S_N (this has a simple geometric explanation). So a natural question is to characterize the partially spherical cyclotomic RCA as a subalgebra of the degenerate DAHA.

Giving such a characterization is one of the goals of this talk. In fact, we will give four alternative definitions of partially spherical cyclotomic RCA.

2. Partially spherical cyclotomic RCA as a subalgebra of degenerate DAHA given by generators.

Definition 0.2. The degenerate DAHA $H\!\!H_{N,\mathrm{deg}}(\hbar, k)$ is generated by $S_N \ltimes \mathbb{Z}^N$ (generated by s_i and invertible commuting elements $X_1, ..., X_N$) and elements $y_1, ..., y_N$ with commutation relations

$$s_i y_i = y_{i+1} s_i + k,$$
$$[y_i, y_j] = 0,$$
$$[y_i, X_j] = k X_j s_{ij},\ i > j,$$
$$[y_i, X_j] = k X_i s_{ij},\ i < j,$$
$$[y_i, X_i] = \hbar X_i - k \sum_{r<i} X_r s_{ir} - k \sum_{r>i} X_i s_{ir}$$

(and the relations of $S_N \ltimes \mathbb{Z}^N$).

Proposition 0.3. *The algebra* $\mathbf{p}\mathbb{H}\mathbb{H}_N^{l,\mathrm{cyc}}(c, \hbar, k)\mathbf{p}$ *may be realized as the subalgebra* $H\!\!H^l_{N,\mathrm{deg}}(z, \hbar, k)$ *of* $H\!\!H_{N,\mathrm{deg}}(\hbar, k)$ *generated by* S_N, X_i, y_i *and the element*

$$D_1^{(l)} := X_1^{-1}(y_1 - z_1)...(y_1 - z_l),$$

where z_i *are related to* c_j *via the equations*

$$z_i = \frac{1}{l}(\hbar(l-i) + \sum_j c_j \zeta^{ij}). \tag{1}$$

Similar results were obtained by Kodera and Nakajima (for spherical subalgebras) and by Webster.

Definition 0.4. The algebra $H\!\!H^l_{N,\mathrm{deg}}(z, \hbar, k)$ is called the degenerate cyclotomic DAHA.

Note that this definition makes sense also for $l = 0$, in which case we recover the whole degenerate DAHA.

3. Degenerate cyclotomic DAHA as a subalgebra of degenerate DAHA preserving certain spaces of functions.

Let D_i be the *rational Dunkl operators*

$$D_i := \hbar\partial_i - \sum_{j\neq i} \frac{k}{X_i - X_j}(1 - s_{ij}),$$

where ∂_i is the derivative with respect to X_i. Define the *trigonometric Dunkl operators* by the formula

$$D_i^{\mathrm{trig}} := X_i D_i - k \sum_{j<i} s_{ij}.$$

Proposition 0.5. *(Cherednik) We have a representation ρ of $H\!\!H_{N,\deg}(\hbar, k)$ on $\mathbf{P} := \mathbb{C}[X_1^{\pm 1}, ..., X_N^{\pm 1}]$, defined by*

$$\rho(X_i^{\pm 1}) = X_i^{\pm 1},\ \ \rho(s_i) = s_i,\ \ \rho(y_i) = D_i^{\mathrm{trig}}.$$

Proposition 0.6. *Suppose $z_i - z_j$ are not integers. Then for Weil generic k, the algebra $H\!\!H^l_{N,\deg}(z, \hbar, k)$ may be realized as the subalgebra of elements $L \in H\!\!H_{N,\deg}(\hbar, k)$ such that $\rho(L)$ preserves the spaces $(X_1...X_N)^{z_i}\mathbb{C}[X_1, ..., X_N]$ for all $i = 1, ..., l$.*

4. Presentation of degenerate cyclotomic DAHA by generators and relations.

Proposition 0.7. *The degenerate cyclotomic DAHA $H\!\!H^l_{N,\deg}(z,\hbar,k)$ is generated by S_N and y_i, X_i, D_i, $i=1,...,N$, with the following defining relations (where $D_1 = D_1^{(l)}$ and $D_i = s_{1i}D_1s_{1i}$ for $i>1$):*

$$s_iy_i = y_{i+1}s_i + k,$$

$$[y_i, y_j] = 0,$$

$$sX_i = X_{s(i)}s,\ \ s\in S_N,\ \ [X_i, X_j] = 0,$$

$$[y_i, X_1] = kX_1s_{1i},\ \ i>1,$$

$$[y_1, X_1] = \hbar X_1 - k\sum_{i>1} X_1s_{1i},$$

$$sD_i = D_{s(i)}s,\ \ [D_i, D_j] = 0,$$

$$[y_j, D_1] = -ks_{1j}D_1,\ \ j>1,$$

$$[y_1, D_1] = -\hbar D_1 + k\sum_{i>1} s_{1i}D_1,$$

$$[D_1, X_1] =$$

$$\sum_{r=1}^{l}\prod_{i=1}^{r-1}(y_1 - z_i + \hbar - k\sum_{j>1} s_{1j})(\hbar - k\sum_{j>1} s_{1j})\prod_{i=r+1}^{l}(y_1 - z_i),$$

$$[D_1, X_m] =$$

$$k\sum_{r=1}^{l}\prod_{i=1}^{r-1}(y_1 - z_i + \hbar - k\sum_{j>1} s_{1j})s_{1m}\prod_{i=r+1}^{l}(y_1 - z_i), m>1,$$

$$X_1D_1 = (y_1 - z_1)...(y_1 - z_l).$$

5. A geometric construction of degenerate cyclotomic DAHA.

Let $W = \mathbb{C}^l$, $V = \mathbb{C}^N$ with standard basis $v_1, ..., v_N$, $L_i \subset V((u))$, $i \in \mathbb{Z}$, are the subspaces such that $L_0 = V[[u]]$, $L_{j+N} = uL_j$, and

$$L_j = uV[[u]] \oplus \mathbb{C}v_1 \oplus ... \oplus \mathbb{C}v_{N-j},\ j = 0, ..., N-1.$$

Let $\mathcal{R}(N, l)$ be the space of the following data:

(a) a sequence of $\mathbb{C}[[u]]$-lattices $M_i \subset V((u))$, $i \in \mathbb{Z}$, such that $M_i \supsetneq M_{i+1}$ and $M_{j+N} = uM_j$;

(b) a $\mathbb{C}((u))$-linear map $b : V((u)) \to V((u))$; and

(c) a $\mathbb{C}((u))$-linear map $p : W((u)) \to V((u))$;

such that

(1) b strongly preserves L and M, i.e., $bL_i \subset L_{i+1}$ and $bM_i \subset M_{i+1}$; and

(2) $pW[[u]] \subset L_0 \cap M_0$.

Note that if $l = 0$ (i.e., $W = 0$), then (c) and (2) drop out, and $\mathcal{R}(N, l)$ is the affine Steinberg variety.

Let $\mathcal{P}$ be the stabilizer of the affine flag L in the group $GL(V)(\mathbb{C}((u)))$. Then $\mathcal{P}$ acts on $\mathcal{R}(N, l)$. Also we have two actions of $\mathbb{C}^*$ dilating b and u, respectively, and an action of the maximal torus $T(W)$ of $GL(W)$. So we can consider the equivariant Borel-Moore homology

$$H_\bullet^{\mathbb{C}^* \times T(W) \times \mathcal{P} \rtimes \mathbb{C}^*}(\mathcal{R}(N, l)).$$

This homology carries an algebra structure constructed using convolution, similarly to how it's done for the usual and affine Steinberg variety.

Proposition 0.8. *The algebra*

$$H_\bullet^{\mathbb{C}^*\times T(W)\times\mathcal{P}\rtimes\mathbb{C}^*}(\mathcal{R}(N,l))$$

is isomorphic to the degenerate cyclotomic DAHA $\mathbb{H}\mathbb{H}^l_{N,\deg}$. *Namely, the equivariant parameter for the loop rotation is* $\hbar$, *the equivariant parameter for dilation of* b *is* k, *the equivariant parameters for* $T(W)$ *are* z_i *and for* $\mathcal{P}$ *are the elements* y_i *(which are not central).*

Note that for $l = 0$ this recovers the result of Oblomkov and Yun, saying that degenerate DAHA may be realized as the equivariant Borel-Moore homology of the affine Steinberg variety.

6. Cyclotomic DAHA. Now consider the q-deformed situation. We have q-deformations of all the four realizations. We start with the realization as a subalgebra in DAHA given by generators. We first recall the definition of DAHA.

Definition 0.9. (Cherednik) The DAHA $\mathbb{H}\mathbb{H}_N(q,t)$ is generated by invertible elements X_i, Y_i, $i = 1, ..., N$, and T_i, $i = 1, ..., N-1$, with relations

$$(T_i - \mathbf{t})(T_i + \mathbf{t}^{-1}) = 0,$$
$$T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1},$$
$$T_iT_j = T_jT_i \ (|i-j| \geq 2),$$
$$T_iX_iT_i = X_{i+1},$$
$$T_iX_j = X_jT_i \ (j \neq i, i+1),$$
$$T_iY_iT_i = Y_{i+1},$$
$$T_iY_j = Y_jT_i \ (j \neq i, i+1),$$
$$X_1T_1Y_1 = T_1Y_1T_1X_1T_1.$$
$$Y_i\tilde{X} = q\tilde{X}Y_i,$$
$$X_i\tilde{Y} = q^{-1}\tilde{Y}X_i,$$
$$[X_i, X_j] = 0,$$
$$[Y_i, Y_j] = 0.$$

where $\tilde{X} := \prod_i X_i$, $\tilde{Y} = \prod_i Y_i$, $t = \mathbf{t}^2$.

Definition 0.10. The cyclotomic DAHA $\mathbb{H}\mathbb{H}_N^l(Z,q,t)$, $Z = (Z_1, ..., Z_l)$, is the subalgebra of DAHA generated by $T_j, X_i, Y_i^{\pm 1}$, and the element

$$D_1^{(l)} := X_1^{-1}(Y_1 - Z_1)...(Y_1 - Z_l).$$

7. Definition of cyclotomic DAHA as the subalgebra preserving certain spaces of functions.

Proposition 0.11. *(Cherednik) We have an action of $\mathbb{H}\mathbb{H}_N(q,t)$ on $\mathbf{P}$ given by*

$$\rho(X_i) = X_i,$$

$$\rho(T_i) = \mathbf{t}s_i + \frac{\mathbf{t}-\mathbf{t}^{-1}}{X_i/X_{i+1}-1}(s_i - 1),$$

$$\rho(Y_i) = \mathbf{t}^{N-1}\rho(T_i^{-1}...T_{N-1}^{-1})\omega\rho(T_1...T_{i-1}),$$

where $(\omega f)(X_1, ..., X_N) := f(qX_N, ..., X_{N-1})$.

Let $q = e^{\varepsilon\hbar}$, $t = e^{-\varepsilon k}$, $Z_i = q^{z_i}$, where ε is a formal parameter.

Proposition 0.12. *For $z_i - z_j \notin \mathbb{Z}$ and k Weil generic, $\mathbb{H}\mathbb{H}_N^l(Z,q,t)$ may be characterized as the subalgebra of elements L of $\mathbb{H}\mathbb{H}_N(q,t)$ such that $\rho(L)$ preserves the spaces $(X_1...X_N)^{z_i}\mathbb{C}[X_1,...,X_N]$ for $i = 1,...,l$.*

8. Presentation of cyclotomic DAHA by generators and relations.

Theorem 0.13. *The subalgebra* $\mathbb{H}\mathbb{H}_N^{l,+} \subset \mathbb{H}\mathbb{H}_N^l$ *generated by* T_i, X_i, $D_i := D_i^{(l)}$, *and* Y_i *has the following defining relations:*

$$(T_i - \mathbf{t})(T_i + \mathbf{t}^{-1}) = 0;$$
$$T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1};$$
$$T_iT_j = T_jT_i \ (|i-j| \geq 2);$$
$$T_iX_iT_i = X_{i+1};$$
$$T_iX_j = X_jT_i \ (j \neq i, i+1);$$
$$T_iY_iT_i = Y_{i+1};$$
$$T_iY_j = Y_jT_i \ (j \neq i, i+1);$$
$$[X_i, X_j] = 0;$$
$$[Y_i, Y_j] = 0;$$
$$X_iY_j = Y_jX_iT_{j-1}^{-1}...T_{i+1}^{-1}T_i^2T_{i+1}...T_{j-1}, \ i < j;$$
$$Y_iX_j = T_{j-1}^{-1}...T_{i+1}^{-1}T_i^2T_{i+1}...T_{j-1}X_jY_i, \ i < j;$$
$$Y_iT_{i-1}^{-1}...T_1^{-2}...T_{i-1}^{-1}X_i = qX_iT_i...T_{N-1}^2...T_iY_i;$$
$$[D_i, D_j] = 0;$$
$$T_i^{-1}D_iT_i^{-1} = D_{i+1}, \ [T_j, D_i] = 0 \ for \ |i-j| \geq 2;$$
$$D_iY_j = Y_jT_{j-1}^{-1}....T_{i+1}^{-1}T_i^{-2}T_{i+1}...T_{j-1}D_i, \ i < j;$$
$$D_jT_{j-1}...T_{i+1}T_i^2T_{i+1}^{-1}...T_{j-1}^{-1}Y_i = Y_iD_j, \ i < j;$$
$$D_iY_iT_{i-1}^{-1}...T_1^{-2}...T_{i-1}^{-1} = T_i...T_{N-1}^2...T_iY_iD_i;$$
$$X_1D_1 = (Y_1 - Z_1)...(Y_1 - Z_l);$$
$$D_1X_1 = (qJ_NY_1 - Z_1)...(qJ_NY_1 - Z_l);$$
$$[D_1, X_2] =$$
$$(\mathbf{t}^{-1} - \mathbf{t})\sum_{r=1}^{l}(qJ_NY_1 - Z_1)...(qJ_NY_1 - Z_{r-1})Y_2T_1^{-2}(Y_2T_1^{-2} - Z_{r+1})...(Y_2T_1^{-2} - Z_l)T_1.$$

where $J_N = T_1...T_{N-1}^2...T_1$. *The algebra* $\mathbb{H}\mathbb{H}_N^l$ *is defined by the same generators and relations, adding the condition that* Y_i *are invertible.*

Corollary 0.14. *(the PBW theorem for cyclotomic DAHA)*

(i) Ordered monomials in X_i *and* D_i *which miss either* X_i *or* D_i *for each* i *form a basis of* $\mathbb{H}\mathbb{H}_N^{l,+}$ *as a left or right module over the positive part of the affine Hecke algebra* H_N^+ *generated by* $T_s, s \in S_N$ *and* Y_i*, and a basis of* $\mathbb{H}\mathbb{H}_N^l$ *as a left or right module over the affine Hecke algebra* H_N *generated by* $T_s, s \in S_N$ *and* $Y_i^{\pm 1}$*; in particular,* $\mathbb{H}\mathbb{H}_N^{l,+}$ *is a free module over* H_N^+ *and* $\mathbb{H}\mathbb{H}_N^l$ *is a free module over* H_N*.*

(ii) $\mathbb{H}\mathbb{H}_N^{l,+}$ *is a free module over the polynomial algebra* $\mathbb{C}[X_1, ..., X_N] \otimes \mathbb{C}[D_1, ..., D_N]$ *of rank* $N!l^N$*, where the first factor acts by left multiplication and the second one by right multiplication.*

Also we obtain new quantum integrable systems. Namely, let e_i be elementary symmetric functions. Then the elements $M_i := e_i(D_1, ..., D_N)$ act on $\mathbf{P}^{S_N}$ by commuting q-difference operators, which depend on $q, t, z_1, ..., z_N$. Joint eigenfunctions of these operators may be obtained from joint eigenfunctions of Macdonald operators by a certain integral transform.

9. Example: l=1

Theorem 0.15. *(i) $H\!\!H_N^{1,+}(q,t)$ is generated by T_i, X_j, and D_j with the defining relations*

$$(T_i - \mathbf{t})(T_i + \mathbf{t}^{-1}) = 0,$$
$$T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1},$$
$$T_i T_j = T_j T_i \ (|i-j| \geq 2),$$
$$T_i X_i T_i = X_{i+1},\ [T_i, X_j] = 0 \ \textit{for}\ j \neq i, i+1.$$
$$D_i = T_i D_{i+1} T_i,\ [T_i, D_j] = 0 \ \textit{for}\ j \neq i, i+1.$$
$$[X_i, X_j] = 0,$$
$$[D_i, D_j] = 0,$$
$$X_i D_j = D_j T_{j-1} ... T_{i+1} T_i^2 T_{i+1}^{-1} ... T_{j-1}^{-1} X_i$$
$$+(\mathbf{t} - \mathbf{t}^{-1}) T_{j-1}^{-1} ... T_i^{-1} ... T_{j-1}^{-1},\ i < j,$$
$$D_j X_i = X_i T_{i-1}^{-1} ... T_{j+1}^{-1} T_j^{-2} T_{j+1} ... T_{i-1} D_j$$
$$-(\mathbf{t} - \mathbf{t}^{-1}) T_{i-1} ... T_j ... T_{i-1},\ i > j,$$
$$D_1 X_1 + 1 = q J_N (X_1 D_1 + 1).$$

(ii) (the PBW theorem) For any values of parameters, the elements $\prod_i X_i^{m_i} \cdot T_w \cdot \prod_i D_i^{n_i}$ form a basis of $H\!\!H_N^{1,+}(q,t)$.

(iii) The algebra $H\!\!H_N^{1}(q,t)$ is obtained from $H\!\!H_N^{1,+}(q,t)$ by inverting the element $Y_1 := 1 + X_1 D_1$.

Note that as $q \to 1$, this degenerates to the standard presentation of the rational Cherednik algebra for S_N (namely, $\frac{D_i}{q-1}$ tends to classical rational Dunkl operators).

10. The geometric construction of cyclotomic DAHA.

The following result has been proved in the formal setting, but is expected to hold non-formally as well.

Proposition 0.16. *The equivariant K-theory*

$$K^{\mathbb{C}^* \times T(W) \times \mathcal{P} \rtimes \mathbb{C}^*}(\mathcal{R}(N,l))$$

has a natural algebra structure, and is isomorphic to the cyclotomic DAHA $\mathbb{H}^l_N$. Namely, the equivariant parameter for the loop rotation is q, the equivariant parameter for dilation of b is t, the equivariant parameters for $T(W)$ are Z_i and for $\mathcal{P}$ are the elements Y_i (which are not central).

Note that for $l = 0$ we recover the result of Varagnolo and Vasserot, saying that DAHA may be realized as the equivariant K-theory of the affine Steinberg variety.

11. Application: flatness of q-deformed quasiinvariants.

Let $m \in \mathbb{Z}_+$. Let $f \in \mathbb{C}[X_1, ..., X_N]$.

Definition 0.17. (i) (O. Chalykh-A. Veselov) f is an m-quasiinvariant if for any $i < j$ the polynomial $(1 - s_{ij})f$ is divisible by $(X_i - X_j)^{2m+1}$.

(ii) (O. Chalykh) f is a q-deformed m-quasiinvariant if for any $i < j$ the polynomial $(1 - s_{ij})f$ is divisible by $\prod_{r=-m}^{m}(X_i - q^r X_j)$.

Let Q_m be the algebra of m-quasiinvariants, and $Q_{m,q}$ the algebra of q-deformed m-quasiinvariants.

Theorem 0.18. *Any m-quasiinvariant can be q-deformed. In other words, $Q_{m,q}$ is a flat deformation of Q_m (has the same Hilbert series as Q_m).*

Proof. Q_m is a module over the spherical rational Cherednik algebra $\mathbf{e}\mathbb{H}_N^{\mathrm{rat}}(1, m)\mathbf{e}$ (for S_N) from category O. The spherical cyclotomic DAHA $\mathbf{e}H\!\!H_N^{1,+}(q,t)\mathbf{e}$ for $t = q^{-m}$ is a flat deformation of $\mathbf{e}\mathbb{H}_N^{\mathrm{rat}}(1, m)\mathbf{e}$. Moreover, it can be shown that the action of $\mathbf{e}\mathbb{H}_N^{\mathrm{rat}}(1, m)\mathbf{e}$ on Q_m deforms to an action of $\mathbf{e}H\!\!H_N^{1,+}(q,t)\mathbf{e}$ on $Q_{m,q}$. But category O for $\mathbf{e}\mathbb{H}_N^{\mathrm{rat}}(1, m)\mathbf{e}$ is semisimple, so all the modules have full support. Hence, if $Q_{m,q}$ were not a flat deformation, the leading coefficient of the Hilbert polynomial of $Q_{m,q}$ would have been less than that of Q_m. But $Q_{m,q}$ contains the ideal generated by $\prod_{i \neq j} \prod_{r=1}^{m}(X_i - q^r X_j)$, so these coefficients are the same. This implies the statement. □

12. The case $q = 1$.

Consider the spherical subalgebra $\mathbf{e}H\!H^l_N(Z,1,t)\mathbf{e}$. (a commutative domain). Consider the module $H\!H^l_N(Z,1,t)\mathbf{e}$ over this algebra. Let $\mathbb{M}^l_N(Z,t) = \mathrm{Specm}(\mathbf{e}H\!H^l_N(Z,1,t)\mathbf{e})$.

Proposition 0.19. *The algebra $\mathbf{e}H\!H^l_N(Z,1,t)\mathbf{e}$ is finitely generated and Cohen-Macaulay (i.e., $\mathbb{M}^l_N(Z,t)$ is an irreducible Cohen-Macaulay variety) and the module $H\!H^l_N(Z,1,t)\mathbf{e}$ is Cohen-Macaulay. In particular, $H\!H^l_N(Z,1,t)\mathbf{e}$ is projective of rank $N!$ on the smooth locus $\mathbb{M}^l_N(Z,t)_{\mathrm{smooth}}$ of $\mathbb{M}^l_N(Z,t)$.*

(ii) $\mathbb{M}^l_N(Z,t)$ is smooth outside of a set of codimension two and normal.

(iii) The natural map

$$H\!H^l_N(Z,1,t) \to \mathrm{End}_{\mathbf{e}H\!H^l_N(Z,1,t)\mathbf{e}}(H\!H^l_N(Z,1,t)\mathbf{e})$$

is an isomorphism.

(iv) The natural map

$$\mathrm{Center}(H\!H^l_N(Z,1,t)) \to \mathbf{e}H\!H^l_N(Z,1,t)\mathbf{e}$$

given by $\mathbf{z} \mapsto \mathbf{z}\mathbf{e}$ is an isomorphism.

13. Relation to multiplicative quiver varieties.

Let $t \in \mathbb{C}^*$ be not a root of unity, and $Z_1, ..., Z_l \in \mathbb{C}^*$ be such that Z_i/Z_j is not an integer power of t for $i \neq j$. Let Q_l be the cyclic quiver $\hat{A}_{l-1}$ with vertices $1, ..., l$ and an additional "Calogero-Moser vertex" 0 attached to the vertex 1. Let $\mathcal{M}_N^l(Z, t)$ be the *multiplicative quiver variety* for Q_l with dimension vector $d_1 = ... = d_l = N$ and $d_0 = 1$ (Crawley-Boevey, Shaw). Namely, given complex vector spaces V_i, $i = 1, ..., l$, with $\dim V_i = N$, $\mathcal{M}_N^l(Z, t)$ is the variety of collections of linear maps $\mathbf{X}_i : V_{i+1} \to V_i$ and $\mathbf{D}_i : V_i \to V_{i+1}$ (where addition is mod l) satisfying the equations

$$Z_i(1 + \mathbf{X}_i\mathbf{D}_i) = Z_{i-1}(1 + \mathbf{D}_{i-1}\mathbf{X}_{i-1}), 2 \leq i \leq l$$

and

$$Z_1(1 + \mathbf{X}_1\mathbf{D}_1)T = Z_l(1 + \mathbf{D}_l\mathbf{X}_l),$$

where $T : V_1 \to V_1$ is an operator conjugate to $\mathrm{diag}(t^{-1}, ..., t^{-1}, t^{n-1})$, modulo simultaneous conjugation (i.e., the corresponding categorical quotient).

Theorem 0.20. *(i) The variety* $\mathcal{M}_N^l(Z,t)$ *is smooth and connected;*

(ii) We have an isomorphism $\mathcal{M}_N^l(Z,t) \cong \mathbb{M}_N^l(Z,t)$*;*

(ii) The module $H\!H_N^l(Z,1,t)\mathbf{e}$ *over* $\mathrm{Center}(H\!H_N^l(Z,1,t)) \cong \mathbf{e}H\!H_N^l(Z,1,t)\mathbf{e}$ *is projective of rank* $N!$*;*

(iii) $H\!H_N^l(Z,1,t)$ *is a split Azumaya algebra over* $\mathrm{Center}(H\!H_N^l(Z,1,t))$ *of rank* $N!$*, namely the endomorphism algebra of the vector bundle* $H\!H_N^l(Z,1,t)\mathbf{e}$*. Thus, all irreducible representations of* $H\!H_N^l(Z,1,t)$ *have dimension* $N!$ *and are parametrized by points of* $\mathbb{M}_N^l(Z,t)$*.*

Thus, $\mathbf{e}H\!H_N^l(Z,q,t)\mathbf{e}$ is a quantization of the multiplicative quiver variety $\mathcal{M}_N^l(Z,t)$.