Lecture 8: Tensor products.

Bilinear maps. R a ring, M, N, P modules.

$\alpha : M \times N \to P$ *bilinear* if it is linear in each variable. Set of bilinear maps $Bil_R(M, N; P)$.

Tensor product. R a ring, M, N modules.

Their tensor product $M \otimes_R N$, or $M \otimes N$, is the quotient of the free module ~~R Hom(M, N)~~ $R\, M \times N$ by the relations

$$(m+m', n) - (m,n) - (m',n)$$
$$(m, n+n') - (m,n) - (m,n')$$
$$(xm, n) - x(m,n)$$
$$(m, xn) - x(m,n) \quad , \; m, m' \in M,\; n, n' \in N,\; x \in R.$$

> Ex. $\mathbb{Z}/m \otimes_{\mathbb{Z}} \mathbb{Z}/n = \mathbb{Z}/\gcd(m,n)$.
> $\mathbb{Q} \otimes_{\mathbb{Z}} A = \mathbb{Q}^r$.

This yields a canonical bilinear map $\beta : M \times N \to M \otimes_R N$. $\beta(m,n) = m \otimes n$.

Thm. (UMP of tensor product)
$M \otimes_R N$ is universal P for the property of having a bilinear map $M \times N \to P$.
I.e., the map $\mathrm{Hom}(M \otimes_R N, P) \xrightarrow{\beta^*} Bil_R(M, N; P)$ is an isomorphism.

Pf. Clear.

Bifunctoriality: $\otimes_R$ is a functor in each variable that is R-linear on morphisms.

<u>Prop.</u> 1) $M \otimes_R N \xrightarrow{\sim} N \otimes_R M$ by the usual switch map (<u>commutative law</u>)

2) $R \otimes_R M \cong M$. (<u>unitary law</u>)

<u>Pf</u> easy.

<u>Bimodules</u> R, R' rings. An R, R'-bimodule is an R-module M and R'-module s.t. $\forall x \in R, x' \in R', m \in M$ $x(x'm) = x'(xm)$.

Can also think of R-module structure as left and R' as right. Then the bimodule axiom takes the form of associativity: $(xm)x' = x(mx')$

If M is an R-module, N an (R, R')-bimodule then $M \otimes_R N$ is an (R, R')-bimodule.

$(m \otimes n)x' = m \otimes nx'$. E.g. if M is an R-mod then $M \otimes_R R'$ is an R'-module.

(<u>extension of scalars</u>).

$Hom_R(M, N)$ is an (R, R')-bimodule

$Hom_R(N, P)$ ← R-mod is an (R, R')-bimod.

Thm. Let R, R' be rings, M an R-module, N an (R,R')-bimodule, P an R'-module. Then we have two canonical (R,R')-isomorphisms:

1) $M \otimes_R (N \otimes_{R'} P) \longrightarrow (M \otimes_R N) \otimes_{R'} P$ (associative law)

2) $Hom_{R'}(M \otimes_R N, P) \cong Hom_R(M, Hom_{R'}(N,P))$ (adjoint assoc.)

Pf. 1) $Hom_{(R,R')}(M \otimes_R (N \otimes_R P), Q) =$

$= Bil(M, N \otimes_R P; Q) = Tril(M, N, P; Q)$

and same for the other one. So the statement follows from Yoneda lemma.

2) ~~$Hom_{R'}(M \otimes_R N, P) \cong Bil(M, N; P)$~~

Define $\alpha : Hom_{R'}(M \otimes_R N, P) \to Hom_R(M, Hom_{R'}(N, P))$

by $\alpha(\gamma)(m)(n) = \gamma(m \otimes n)$.

Need to check:

1) Well defined (relations of $M \otimes_R N$ hold (this is straightforward).

2) α is invertible. Need to define α^{-1}!

Take $\eta \in Hom_R(M, Hom_{R'}(N, P))$.

Define $\zeta : M \times N \to P$ by $\zeta(m,n) = \eta(m)(n)$.
Then ζ is biadditive, and also $\forall x \in R$

$$\zeta(xm, n) = \eta(xm)(n) = \eta(m)(xn) = \zeta(m, xn),$$

so ζ induces a $\mathbb{Z}$-linear map $\delta : M \otimes_R N \to P$.

~~This descends to a map~~

This map is R'-linear, so set

$\delta = \beta'(\eta)$, and check $\alpha \circ \beta = id$, $\beta \circ \alpha = id$.

<u>Corollary</u> $\boxed{R \to R'}$ 1) $(M \otimes_R R') \otimes_{R'} P = M \otimes_R P$ (cancellation law)

2) $Hom_{R'}(M \otimes_R R', P) = Hom_R(M, P)$

($\otimes_R R'$ is <u>left adjoint</u> to $R'\text{-mod} \to R\text{-mod}$ forgetful f-r.)

3) $Hom_R(M, P) = Hom_{R'}(M, Hom_R(R', P))$,

i.e. $Hom_R(R', P)$ is <u>right adjoint</u> to the forgetful f-r.

<u>Pf.</u> This follows from the above.

<u>Cor</u> If N is an R-module then the functor $\otimes_R N$ on R-mod preserves direct limits, or, equivalently, direct sums and cokernels.

Pf. This is because it's left adjoint.

Rem. Preserving cokernels in particular implies that it's right exact.

Rem. $\otimes_R$ does not preserve kernels (~~kernels~~ (e.g. not left exact).

E.g. $0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z}$, $\mathrm{Ker} = 0$.

But if $\otimes_{\mathbb{Z}} \mathbb{Z}/2$ get $0 \to \mathbb{Z}/2 \xrightarrow{0} \mathbb{Z}/2$, $\mathrm{Ker} = \mathbb{Z}/2$.

Thm. (Watts) $F : R\text{-mod} \to R\text{-mod}$ a linear functor. Then have a natural transf.
$\theta : \bullet \otimes_R F(R) \to F(\bullet)$ s.t. $\theta(R) = 1$, and θ is an isomorphism if and only if F preserves direct limits (= direct sums and cokernels).

Pf. Since F is linear, there is a natural linear map $\theta(M) : \underset{\underset{M}{\|}}{\mathrm{Hom}}(R, M) \to \mathrm{Hom}(F(R), F(M))$.
Set $N = F(R)$. Then setting $P = F(M)$ in adjoint associativity, we get

$\theta(M) \in \mathrm{Hom}(M, \mathrm{Hom}(N, F(M))) = \mathrm{Hom}(M \otimes N, F(M))$.

Namely, $\theta(M)(m \otimes n) = F(\rho)(n)$, where $\rho : R \to M$, $\rho(1) = m$. If θ is an isom, then F preserves direct sums and cokernels.

Conversely, suppose F preserves direct sums and cokernels, and let's show that θ is an isomorphism.

Pick a presentation $R^{\oplus\Sigma} \to R^{\oplus\Lambda} \to M$

Applying θ to this presentation, we get a comm. diagram

$$\begin{array}{ccccccc} R^{\oplus\Sigma}\otimes N & \longrightarrow & R^{\oplus\Lambda}\otimes N & \longrightarrow & M\otimes N & \longrightarrow & 0 \\ \downarrow \theta(R^{\oplus\Sigma}) & & \downarrow \theta(R^{\oplus\Lambda}) & & \downarrow \theta(M) & & \\ F(R^{\oplus\Sigma}) & \longrightarrow & F(R^{\oplus\Lambda}) & \longrightarrow & F(M) & \longrightarrow & 0 \end{array}$$

By construction, $\theta(R) = 1_N$. If F preserves direct sums, $\theta(R^{\oplus\Lambda}) = 1_{N^{\oplus\Lambda}}$, $\theta(R^{\oplus\Sigma}) = 1_{N^{\oplus\Sigma}}$.

~~But~~ (if T, U are functors preserving $\oplus$ and $\theta: T \to U$ a morphism, then it also preserves $\oplus$, i.e. $\theta(\bigoplus_\lambda M_\lambda) = \bigoplus_\lambda \theta(M_\lambda)$).

Since F also preserves cokernels, and so does $\otimes N$, the rows are exact. So $\theta(M)$ is an isomorphism.

<u>Additive functors.</u> An additive functor is a $\mathbb{Z}$-linear functor on R-mod. Then F preserves finite direct sums (exer.).

(Recall that $X \xrightarrow{i_X} X \oplus Y \xrightarrow{p_X} X$, $Y \xrightarrow{i_Y} X\oplus Y \xrightarrow{p_Y} Y$, $\quad p_X i_X = 1$, $p_Y i_Y = 1$, $i_X p_X + i_Y p_Y = 1$.)

Conversely, any functor that preserves

finite direct sums is $\mathbb{Z}$-linear:

if $\alpha, \beta : M \to N$ then $F(\alpha \oplus \beta) = F(\alpha) \oplus F(\beta)$

Also we have $\delta_M : M \to M \oplus M$ (diagonal)

and $\sigma_M : M \oplus M \to M$ (addition) $\sigma_M(m_1 \oplus m_2) = m_1 + m_2$

So $F(\alpha+\beta) = F(\alpha) + F(\beta)$.

But not every $\mathbb{Z}$-linear functor is R-linear. E.g. Complex conj. functor on the category of $\mathbb{C}$-vector spaces. $V \mapsto \overline{V}$ where $\overline{V}$ is V with twisted scalar mult. $F(\alpha) = \alpha$. Then $F(\lambda\alpha)(v) = \lambda\alpha v$, $\lambda F(\alpha)(v) = \overline{\lambda}\alpha v$, so $F(\lambda\alpha) \neq \lambda F(\alpha)$.

<u>Lemma</u>. M, N R-modules, n_λ gen. of N. Then $\forall t \in M \otimes N$ can be written as $t = \sum m_\lambda \otimes n_\lambda$ with $m_\lambda \in M$, almost all 0. Further, $t = 0 \iff \exists\, p_\sigma \in M, x_{\sigma\lambda} \in R$ $(\sigma \in \Sigma)$ with $m_\lambda = \sum x_{\lambda\sigma} p_\sigma$, and $\sum x_{\lambda\sigma} n_\lambda = 0$ $\forall \sigma$.

<u>Pf</u>. $M \otimes N$ is generated by $m \otimes n$, and $n = \sum x_\lambda n_\lambda$, so $m \otimes n = \sum m x_\lambda \otimes n_\lambda$. So the sum repr. exists.

Supp. $p_\sigma, x_{\lambda\sigma}$ exist, then clearly $t = 0$.

Conversely, supp. $t = 0$. ~~Then~~

Pick a presentation $R^{\oplus\Sigma} \xrightarrow{\beta} R^{\oplus\Lambda} \xrightarrow{\alpha} N \to 0$ with $e_\lambda \mapsto n_\lambda$.

So we have an exact sequence

$$M \otimes R^{\oplus \Sigma} \xrightarrow{1 \otimes \beta} M \otimes R^{\oplus \Lambda} \xrightarrow{1 \otimes \alpha} M \otimes N \to 0$$

Now $(1 \otimes \alpha)(\sum m_\lambda \otimes e_\lambda) = 0$. Hence by exactness $\exists\, s \in M \otimes R^{\Sigma}$ s.t. $(1 \otimes \beta)(s) = \sum m_\lambda \otimes e_\lambda$.

Let $\{e_\sigma\}$ be the standard basis of $R^{\oplus \Sigma}$, and write $s = \sum p_\sigma \otimes e_\sigma$, $p_\sigma \in M$. Also $\beta(e_\sigma) = \sum x_{\lambda\sigma} e_\lambda$. Then $0 = \alpha\beta(e_\sigma) = \sum_\lambda x_{\lambda\sigma} n_\lambda$, and

$$0 = \underbrace{\sum m_\lambda \otimes e_\lambda}_{s} - \sum p_\sigma \underbrace{\otimes}_{s} \left(\sum_\lambda x_{\lambda\sigma} e_\lambda\right) =$$

$$= \sum \left(m_\lambda - \sum_\sigma x_{\lambda\sigma} p_\sigma\right) \otimes e_\lambda$$

$$\Rightarrow m_\lambda = \sum_\sigma x_{\lambda\sigma} p_\sigma.$$

S, T R-algebras $\Rightarrow$ $S \otimes T$ is an R-algebra with $(s_1 \otimes t_1)(s_2 \otimes t_2) = s_1 s_2 \otimes t_1 t_2$ (check that it's well def.)

Also, if V is an R-algebra and have $s : S \to V$, $t : T \to V$ algebra homom then have $s \otimes t : S \otimes T \to V$. This gives a bijection

$$\mathrm{Hom}_{R\text{-alg}}(S \otimes T, V) \longrightarrow \mathrm{Hom}_{R\text{-alg}}(S, V) \times \mathrm{Hom}_{R\text{-alg}}(T, V).$$

In other words, $S \otimes_R T$ is the <u>coproduct</u> of S, T in R-alg.

Ex. S an R-alg, $X_1, \ldots, X_n$ variables.

$$S \underset{R}{\otimes} R[X_1, \ldots, X_n] = S[X_1, \ldots, X_n].$$

$$R[X_1, \ldots, X_n] \underset{R}{\otimes} R[Y_1, \ldots, Y_m] = R[X_1, \ldots, X_n, Y_1, \ldots, Y_m].$$