Lecture 7.

Filtered direct limits.

Λ a small category (diagram)

Def. Λ filtered if 1) $\forall \varkappa, \lambda \in \Lambda\ \exists \mu$ $\begin{matrix} \varkappa \to \mu \\ \lambda \nearrow \end{matrix}$

2) given $\sigma, \tau : \eta \rightrightarrows \varkappa$ $\exists\ \varphi : \varkappa \to \mu$

that coequalizes them: $\varphi \circ \sigma = \varphi \circ \tau$.

Given a category $\mathcal{C}$, say $\lambda \to M_\lambda$ filtered if Λ filtered. Then we say $\varinjlim M_\lambda$ filtered if exists.

Ex: Union is a filtered direct limit.

More generally:

Ex. Λ a poset. Supp Λ directed: $\forall \varkappa, \lambda\ \exists \mu$ $\varkappa \le \mu, \lambda \le \mu$. Let $Hom(\varkappa, \lambda)$ cons. of a single elt if $\varkappa \le \lambda$, and empty if not. This is a filtered cat.

Ex. $\Lambda = \mathbb{Z}_+$, $\forall n \in \Lambda$ $M_n = \{\frac{r}{n}, r \in \mathbb{Z}\}$. Then $\varinjlim_{\mathbb{Z}_+} M_n = \mathbb{Q}$. (ordered by divisibility).

We can also identify M_n with $\mathbb{Z}$ and write $\varinjlim \mathbb{Z} = \mathbb{Q}$.

Thm. If Λ is a filtered category, R a rig, $\mathcal{C}$ sets or R-mod or R-alg, $\lambda \mapsto M_\lambda$ functor $\Lambda \to \mathcal{C}$.

Define a relation $\sim$ on the set-theoretic union $\coprod_\lambda M_\lambda$ as follows: $\overset{\in M_{\lambda_1}}{m_1} \sim \overset{\in M_{\lambda_2}}{m_2}$

if $\exists\, \alpha_i : M_{\lambda_i} \to M_\mu$ (coming from Λ)

s.t. $\alpha_1(m_1) = \alpha_2(m_2)$. Then $\sim$ is an equiv. relation. Set $M = \coprod M_\lambda / \sim$. Then $M = \varinjlim M_\lambda$, and for each μ, the canonical map $\alpha_\mu : M_\mu \to M$ is the structure map $M_\mu \to \varinjlim M_\lambda$

Pf. Clearly $\sim$ refl. and sym. Show it's transitive

Supp. $m_1 \sim m_2$, $m_2 \sim m_3$, so $\alpha_1(m_1) = \alpha_2(m_2)$

$\alpha_2'(m_2) = \alpha_3'(m_3)$

$$\begin{tikzcd}
\lambda_1 \arrow[dr, "\alpha_1"] & & & \\
 & \mu \arrow[r, "\beta"] & \nu \arrow[r, "\gamma"] & x \\
\lambda_2 \arrow[ur, "\alpha_2"] \arrow[dr, "\alpha_2'"'] & & & \\
 & \mu' \arrow[uur, "\beta'"'] & & \\
\lambda_3 \arrow[ur, "\alpha_3"'] & & &
\end{tikzcd}$$

equalize $\beta\alpha_1$ and $\beta'\alpha_3''$.

If we are in R-mod, we also have to see why $M \in R$-mod.

Define addition on M $\forall\, m_i \in M_{\lambda_i}$, pick α_1, α_2 ~~$\alpha_1(m_1) = \alpha_2(m_2)$ and take~~ $\lambda_1 \to \mu$, $\lambda_2 \to \mu$, and

set $\alpha_{\lambda_1} m_1 + \alpha_{\lambda_2} m_2 := \alpha_\mu(\alpha_1(m_1) + \alpha_2(m_2))$.

Easy to check this is well defined.

Scalar mult. is defined similarly.

Also if ~~Ms are~~ R-algebras can define mult. similarly.

The rest is easy.

<u>Cor.</u> (1) $\forall m \in \varinjlim M_\lambda$ $\exists \lambda$ and $m_\lambda \in M_\lambda$ s.t. $m = \alpha_\lambda(m_\lambda)$.

(2) $\forall m_i \in M_{\lambda_i}$ for $i=1,2$, s.t. $\alpha_{\lambda_1}(m_1) = \alpha_{\lambda_2}(m_2)$ $\exists \mu$ s.t. $\exists \alpha_1 : \lambda_1 \to \mu$, $\alpha_2 : \alpha_2 \to \mu$ with $\alpha_1(m_1) = \alpha_2(m_2)$ in M_μ.

(3) If $\mathcal{C} = R$-mod or R-alg. If $\alpha_\lambda(m_\lambda) = 0$ then $\exists \mu$ and $\alpha : \lambda \to \mu$ s.t. $\alpha(m_\lambda) = 0$.

<u>Def.</u> R' fin. pres. R-algebra if: $R' = R[X_1, \dots, X_n]/I$, where I is f.gen.

<u>Prop.</u> Λ a filtered category, R a ring, $\mathcal{C} = R$-mod or R-alg, $\lambda \to M_\lambda$ functor $\Lambda \to \mathcal{C}$. Consider the map

$$\theta : \varinjlim \operatorname{Hom}(N, M_\lambda) \longrightarrow \operatorname{Hom}(N, \varinjlim M_\lambda).$$

(1) If N is fin. gen. then θ is injective.

(2) The following conditions are equivalent:

(a) N is finitely presented

(b) θ is bijective for all Λ and all $\Lambda \to \mathcal{C}$;

(c) θ is surjective for all directed sets Λ and all $\lambda \mapsto M_\lambda$.

Finite presentations. Let R be a ring, R' a f.pres. algebra. Then for any presentation $R' = R[X_1, \ldots, X_n]/\mathfrak{a}$ of R', where $R[X_1, \ldots, X_n]$ is a pol. ring and $\mathfrak{a}$ an ideal, $\mathfrak{a}$ is always fin. gen. (this generalizes a statement about fin. presented modules from a prev. lecture).

Exactness of filtered direct limits.

R ring, Λ a filtered category. $\mathcal{C}$ category of 3-term exact sequences of R-modules. Then for any functor $\lambda \mapsto (L_\lambda \xrightarrow{\beta_\lambda} M_\lambda \xrightarrow{\gamma_\lambda} N_\lambda)$ the induced sequence $\varinjlim L_\lambda \xrightarrow{\beta} \varinjlim M_\lambda \xrightarrow{\gamma} \varinjlim N_\lambda$ is exact.

Pf. Given $l \in \varinjlim L_\lambda$, $\exists\, l_\lambda \in L_\lambda$ with $\alpha_\lambda(l_\lambda) = l$. By hypothesis, $\gamma_\lambda \beta_\lambda l_\lambda = 0$, so $\gamma\beta l = 0$. So $\operatorname{Im}\beta \subset \operatorname{Ker}\gamma$.

For the opposite inclusion, let $m \in \varinjlim M_\lambda$ with $\gamma(m) = 0$. $\exists\, m_\lambda$ s.t. $\alpha_\lambda m_\lambda = m$.

~~So~~ Now $\alpha_\lambda \gamma_\lambda m_\lambda = \gamma \alpha_\lambda m_\lambda = \gamma m = 0$.

$$\begin{tikzcd} L_\lambda \arrow[r, "\beta_\lambda"] \arrow[d, "\alpha_\lambda"'] & M_\lambda \arrow[r, "\gamma_\lambda"] \arrow[d, "\alpha_\lambda"] & N_\lambda \arrow[d, "\alpha_\lambda"] \\ L \arrow[r, "\beta"] & M \arrow[r, "\gamma"] & N \end{tikzcd}$$

Thus, we have a transition map $\alpha : \lambda \to \mu$ s.t. $\alpha(\gamma_\lambda m_\lambda) = 0$.
So $\gamma_\mu \alpha m_\lambda = 0$ Hence $\exists\, l_\mu$ s.t. $\alpha m_\lambda = \beta_\mu(l_\mu)$. Apply α_μ, get

$$\beta \alpha_\mu l_\mu = \alpha_\mu \beta_\mu l_\mu = \alpha_\mu \alpha m_\lambda = \alpha_\lambda m_\lambda = m.$$

So $\operatorname{Ker}\gamma \subset \operatorname{Im}\beta$. Hence $\operatorname{Ker}\gamma = \operatorname{Im}\beta$ as asserted.

Hom and direct limits.

Λ a filtered category, R a ring, N a module, $\lambda \mapsto M_\lambda$ functor $\Lambda \to R$-mod. Here is another pf that $\varinjlim \operatorname{Hom}(N, M_\lambda) \to \operatorname{Hom}(N, \varinjlim M_\lambda)$ is injective if N is fin. gen. and bijective if N is finitely presented.

If $N = R$ then θ_N is bijective obviously. Assume that N is fin. gen, and take a presentation $R^\Sigma \to R^n \to N \to 0$. with Σ finite if N is finitely presented.
Then we have the following comm. diag:

$$\begin{tikzcd}
0 \arrow[r] & \varinjlim \operatorname{Hom}(N, M_\lambda) \arrow[r] \arrow[d, "\theta_N"] & \varinjlim \operatorname{Hom}(R^n, M_\lambda) \arrow[r] \arrow[d, "\theta_{R^n}"] & \varinjlim \operatorname{Hom}(R^\Sigma, M_\lambda) \arrow[d, "\theta_{R^\Sigma}"] \\
0 \arrow[r] & \operatorname{Hom}(N, \varinjlim M_\lambda) \arrow[r] & \operatorname{Hom}(R^n, \varinjlim M_\lambda) \arrow[r] & \operatorname{Hom}(R^\Sigma, \varinjlim M_\lambda)
\end{tikzcd}$$

The rows are exact since Hom is left exact, and because of exactness of filtered direct limits.
But Hom preserves finite direct sums, and so does direct limits (commutativity of two direct limits), so θ_{R^n} is bijective, and θ_Σ is bijective if Σ is finite. Now a diagram chase yields the assertion.

Counterexample:

1) N inf. gen, θ_N not injective:

$M_1 = M_2 = \cdots = \mathbb{C}[x]$ $\Lambda = \circ \longrightarrow \circ \longrightarrow \circ \longrightarrow$

$M_i \xrightarrow{\partial} M_{i+1} =$ $\partial = \frac{d}{dx}$. Then $\varinjlim M_i = 0$.

But if $N = \langle e_1, e_2, \ldots \rangle$ Then

$\lim \mathrm{Hom}(N, M_\lambda) \neq 0$.

2) $N = \mathbb{C}[x_1, \ldots, x_n, \ldots] / (x_1, \ldots, x_n, \ldots)$ $\lhd R$

not fin. presented.

$M_1 = R/(x_1)$, $M_2 = R/(x_1, x_2)$. ... θ_N not surjective.

$\varinjlim M_i = \mathbb{C}$. But $\mathrm{Hom}(N, \lim M_i) \neq 0$, $\mathrm{Hom}(N, M_i) = 0$.