Lecture 6: Direct limits

Categories. A category $\mathcal{C}$ is a collection of objects and morphisms (maps). for each $A, B \in \mathcal{C}$, have a set $Hom_{\mathcal{C}}(A,B)$ of morphisms.

Comp. law:

$Hom_{\mathcal{C}}(A,B) \times Hom_{\mathcal{C}}(B,C) \to Hom_{\mathcal{C}}(A,C)$.

$1_B \in Hom_{\mathcal{C}}(B,B)$

Axioms: associativity, unit axiom.

$\alpha : A \to B$ isomorphism with inverse β if

$\alpha\beta = 1_B$, $\beta\alpha = 1_A$.

Ex: Sets, Rings, R-mod, R-alg.

Product of categories: $\mathcal{C}_1 \times \mathcal{C}_2$ (obvious def.).

Functors: (covariant): $A \in \mathcal{C} \longmapsto F(A) \in \mathcal{C}'$

$F : \mathcal{C} \to \mathcal{C}'$

assigns to each $\alpha : A \to B \in \mathcal{C}$ $F(\alpha) : F(A) \to F(B)$ in $\mathcal{C}'$ which satisfies $F(1_A) = 1_{F(A)}$ and preserves composition. Functors preserve isomorphisms.

Forgetful functor: R-mod $\to$ Sets

Natural transf: Functors $\mathcal{C} \to \mathcal{C}'$ themselves form a category. $\theta : F \to F'$ is a collection of morphisms $\theta(A) : F(A) \to F'(A)$ in $\mathcal{C}'$

such that for each $A, B \in \mathcal{C}$ and any $\alpha : A \to B$ the diagram

$$\begin{array}{ccc} F(A) & \xrightarrow{F(\alpha)} & F(B) \\ \downarrow \theta(A) & & \downarrow \theta(B) \\ F'(A) & \to & F'(B) \end{array}$$

is commutative.

(also called morphism of functors)

<u>Isomorphism of functors</u>: natural transf. $\theta : F \to F'$ s.t. there is an inverse $\theta^{-1} : F' \to F$ s.t. $\theta \circ \theta' = 1_{F'}$, $\theta' \circ \theta = 1_F$.

A <u>contravariant</u> functor $G : \mathcal{C} \to \mathcal{C}'$: same def but $G(\alpha\beta) = G(\beta)G(\alpha)$ and if $\alpha : A \to B$ then $G(\alpha) : G(B) \to G(A)$.

E.g. $Hom(\cdot, N)$ is a contravariant functor from R-mod to R-mod.

<u>Rem.</u> Opposite category: reverse arrows. We can view contravariant functors $F : \mathcal{C} \to \mathcal{C}'$ as covariant functors $F : \mathcal{C} \to \mathcal{C}'^{op}$ or $\mathcal{C}^{op} \to \mathcal{C}'$.

<u>Adjoint functors.</u> $F : \mathcal{C} \to \mathcal{D}$, $G : \mathcal{D} \to \mathcal{C}$ covariant functors. We say that (F, G) is an adjoint pair, F is a left adjoint to G, and G a right adjoint to F if for every pair of objects $A \in \mathcal{C}$ and $B \in \mathcal{D}$ there is given a natural bijection

$$\mathrm{Hom}_{\mathcal{D}}(F(A), B) \cong \mathrm{Hom}_{\mathcal{C}}(A, G(B)).$$

Natural means that it's functorial with respect to A, B.

If adjoint exists, it is unique up to a canonical isomorphism.

Indeed, suppose F_1, F_2 are two left adjoints of G.

Given $A \in \mathcal{C}$, let $\theta(A) : F_1(A) \to F_2(A)$ be the image of $1_{F(A)}$ under the adjoint bijections

$$\mathrm{Hom}_{\mathcal{D}}(F_1(A), F_1(A)) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(A, GF_1(A)) \xrightarrow{\sim}$$
$$\mathrm{Hom}_{\mathcal{D}}(F_2(A), F_1(A))$$

This is natural in A.

Similarly one can construct $\theta' : F_2 \to F_1$, and it is easy to show that $\theta \circ \theta' = \theta' \circ \theta = id$.

<u>Ex.</u> $\Lambda \longrightarrow R\Lambda$ is left adjoint to Sets $\to$ R-modules

$M \longrightarrow M$ as ~~an~~ a set:

$$\mathrm{Hom}_R(R\Lambda, M) = \mathrm{Maps}(\Lambda, M).$$

Similarly, the "polynomial alg" functor

$S \mapsto R[S]$: $\mathrm{Hom}_{R\text{-alg}}(R[S], R') = \mathrm{Hom}_{Sets}(S, R')$

sets $\mapsto$ R-algebras

<u>Direct limits.</u> Assume Λ is small, i.e. objects form a set. Given a functor

$\lambda \mapsto M_\lambda$, $\Lambda \to \mathcal{C}$ (typically a graph $\bullet \to \bullet \to \bullet \to \cdots$)

its <u>direct limit</u> or <u>colimit</u>, denoted $\varinjlim M_\lambda$ or $\varinjlim_{\lambda \in \Lambda} M_\lambda$ is defined to be the object of $\mathcal{C}$ universal among objects P equipped with maps $\beta_\mu : M_\mu \to P$ (<u>insertions</u>) which are compatible with transition maps $M_\lambda \to M_\mu$ coming from Λ. ($\exists!\ \beta : \varinjlim M_\lambda \to P$ s.t.
$$\begin{tikzcd} M_\lambda \arrow[r, "\beta_\lambda"] \arrow[d] & P \\ \varinjlim M_\lambda \arrow[ur, "\beta"'] & \end{tikzcd}$$
)

The functor $\lambda \to M_\lambda$ is called a <u>direct system</u>.

By Yoneda lemma, $\varinjlim M_\lambda$ is unique up to a unique isomorphism, if it exists. We say that $\mathcal{C}$ <u>has direct limits</u> ~~ff~~ indexed by Λ if $\varinjlim M_\lambda$ exists for any M_λ. We say that $\mathcal{C}$ <u>has direct limits</u> if it's true for each small category Λ.

If $F: \mathcal{C} \to \mathcal{D}$, then any direct system M_λ in $\mathcal{C}$ gives a direct system $F(M_\lambda)$ in $\mathcal{D}$. So if both $\varinjlim M_\lambda$ and $\varinjlim F(M_\lambda)$ exist, then by the universal property we have a morphism

$$\phi: \varinjlim F(M_\lambda) \longrightarrow F(\varinjlim M_\lambda)$$

since this is equipped with β^F_μ

If ϕ is always an isomorphism, we say that F <u>commutes with</u> or <u>preserves</u> direct limits.

(this can be established by showing that $F(\varinjlim M_\lambda)$ has UMP).

Assume $\mathcal{C}$ has direct limits indexed by Λ. Then if we have a natural transform of $\lambda \to M_\lambda$ to $\lambda \to N_\lambda$ ($\forall \lambda$ have $\gamma_\lambda: M_\lambda \to N_\lambda$ compatible with Λ).

then we have $\gamma_{univ}: \varinjlim M_\lambda \to \varinjlim N_\lambda$

So $\varinjlim$ is a functor from $\mathrm{Fun}(\Lambda, \mathcal{C})$ to $\mathcal{C}$.

This functor is just the left adjoint of the <u>diagonal functor</u> $\Delta: \mathcal{C} \to \mathrm{Fun}(\Lambda, \mathcal{C})$

$\Delta(M) = \{M_\lambda = M$, all morphisms in Λ go to id$\}$.

$\mathrm{Hom}(\varinjlim M_\lambda, X) = \{\alpha_\lambda : \mathrm{Hom}(M_\lambda, X)$ which are compatible$\}$.

<u>Coproducts</u>. Suppose $\mathcal{C}$ a category, Λ a set, M_λ an object of $\mathcal{C}$ for all $\lambda \in \Lambda$. The <u>coproduct</u> $\coprod_{\lambda\in\Lambda} M_\lambda$, or simply $\coprod M_\lambda$, is the object of $\mathcal{C}$ universal among objects P equipped with a map: $\beta_\mu : M_\mu \to P$ for each $\mu \in \Lambda$.

So if we view Λ as a category with only identity morphisms this is a special case of a direct limit.

<u>Ex.</u> $\Lambda = \emptyset$. Then the coproduct is an object with a unique map to any object P.

E.g. for R-mod it is 0.

<u>Ex.</u> $\bigoplus_\lambda M_\lambda$ in the category of modules is a coproduct. <u>Ex.</u> For sets, disj. union is coproduct.

<u>Coequalizers</u>. Let $\alpha, \alpha' : M \rightrightarrows N$ be two morphisms in category $\mathcal{C}$.

Their coequalizer is defined as the object of $\mathcal{C}$ universal among objects P equipped with a map $\eta : N \to P$ s.t. $\eta\alpha = \eta\alpha'$.

<u>Ex.</u> $\mathcal{C} = R$-mod, coequalizer is $\mathrm{Coker}(\alpha - \alpha')$

In Sets: Take $\sim$ the smallest equiv. rel. s.t. $\alpha(m) \sim \alpha'(m)$ for all $m \in M$.
Then the coequalizer is $N/\sim$ equipped with the quotient map.

Rem. Coequalizer is a special case of direct limit. E.g. let $\Lambda = \underset{1}{\bullet} \rightrightarrows \underset{2}{\bullet}$
$\lambda \to M_\lambda$: $M_1 = M$, $M_2 = N$, the maps go to α and α'. Then coequalizer is $\varinjlim M_\lambda$.

Lemma. A category $\mathcal{C}$ has direct limits if and only if it has coproducts and coequalizers. If $\mathcal{C}$ has direct limits then $F : \mathcal{C} \to \mathcal{C}'$ preserves them iff it preserves coproducts and coequalizers.

Pf. One direction is clear, so we only need to prove the other.
Supp $\mathcal{C}$ has coproducts and coequalizers.
Let Λ be a small category ($\lambda \mapsto M_\lambda$ functor), Σ set of trans. maps $\alpha_\lambda^\mu : M_\lambda \to M_\mu$.

$\forall \sigma \in \Sigma$, let M_σ be the source.
($M_\sigma = M_\lambda$ if $\sigma : M_\lambda \to M_\mu$). Let

$$M = \coprod_{\sigma \in \Sigma} M_\sigma, \text{ and } N = \coprod_{\lambda \in \Lambda} M_\lambda.$$

For each σ there are two maps $M_\sigma \to N$: the inclusion and comp. of inclusion with σ. So we have

$$M \overset{\alpha}{\underset{\alpha'}{\rightrightarrows}} N.$$

Let C be their coequalizer, and $\eta : N \to C$ the corr. map.

We claim that C is the required direct limit. This is easy to check and the remaining statement is easy.

<u>Thm.</u> R-mod and sets have direct limits.

<u>Pf.</u> They have ~~dir~~ coproducts and coequalizers.

<u>Thm.</u> Left adjoint functors preserve direct limits.

<u>Pf.</u> Straightforward.

<u>Prop.</u> $\mathcal{C}$ a cat, Λ, Σ small cat.

If $\mathcal{C}$ has direct limits indexed by Σ then $\mathrm{Fun}(\Lambda, \mathcal{C})$ has direct limits indexed by Σ.

<u>Pf.</u> Left as an exercise.

Thm. (Commutativity of direct limits)

$\Sigma \times \Lambda \longrightarrow \mathcal{C}$. $(\sigma, \lambda) \to M_{\sigma,\lambda}$

Then $\varinjlim_{\sigma} \varinjlim_{\lambda} M_{\sigma,\lambda} = \varinjlim_{\lambda} \varinjlim_{\sigma} M_{\sigma,\lambda}$.

Cor. If $\mathcal{C}$ = Sets of R-mod then $\varinjlim : \mathrm{Fun}(\Lambda, \mathcal{C}) \longrightarrow \mathcal{C}$ preserves coproducts and coequalizers.