Lecture 5: Exact sequences

Def. \( 0 \rightarrow M_{i-1} \xrightarrow{\alpha_i} M_i \xrightarrow{\beta_i} M_{i+1} \rightarrow \cdots \) sequence of module homomorphisms is exact at \( M_i \) if \( \ker \alpha_i = \text{Im} \beta_{i-1} \).

Exact \( \Rightarrow \) exact at all terms (except source and target).

Ex. \( 0 \rightarrow L \xrightarrow{\alpha} M \) is exact \( \iff \) \( L \xrightarrow{\beta} M \) is injective.

\( M \xrightarrow{\beta} N \rightarrow 0 \) exact \( \iff \) \( M \xrightarrow{\beta} N \) surjective.

\[ 0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0 \] exact \( \iff \) \( L = \ker \beta \).

\( L \rightarrow M \rightarrow N \rightarrow 0 \) exact \( \iff \) \( N = \text{Coker} \alpha \).

\[ 0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0 \] exact \( \iff \) \( L = \ker \beta \) and \( \beta \) is injective \( \iff \) \( N = \text{Coker} \alpha \) and \( \alpha \) is injective.

If so, it's called a short exact sequence.

In this case \( \text{LCM} \), \( N = M/L \). So such sequence correspond to extensions \( M \) on \( N \) by \( L \).

Ex. \( 0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0 \)

split short exact sequence

Prop. \( \forall \lambda \in \Lambda \), let \( M'_\lambda \rightarrow M _\lambda \rightarrow M'' \). If each of these is exact, then

\[ \bigoplus M'_\lambda \rightarrow \bigoplus M _\lambda \rightarrow \bigoplus M'' \]

are exact, and vice versa.

Pf. Easy.

Prop. \( 0 \rightarrow M' \xrightarrow{\alpha} M \xrightarrow{\beta} M'' \) a short exact sequence. \( N \in M, N' = \alpha^{-1} (N), \) \( N'' = \beta (N) \). Then \( 0 \rightarrow N' \rightarrow N \rightarrow N'' \rightarrow 0 \) is short exact.

Pf. Easy.
Def. \( p: M \rightarrow M' \) retraction of \( \alpha: M' \rightarrow M \) if \( \beta \circ \alpha = \text{id}_{M'} \). Then \( \alpha \) is injective and \( p \) surj.

Also \( \sigma: M'' \rightarrow M \) a section of \( \beta: M \rightarrow M'' \) if \( \beta \circ \sigma = \text{id}_{M} \). Then \( \beta \) is surjective and \( \sigma \) injective. \( M' \rightarrow M \rightarrow M'' \) splits if \( M = M' \oplus M'' \) with

Prop. Let \( M \rightarrow M' \rightarrow M'' \) be a 3-term exact sequence. TFAE:

1) The sequence splits.

2) Have a retraction \( M \rightarrow M' \) of \( \alpha \), and \( p \) is surj.

3) Have a section \( M'' \rightarrow M \) of \( \beta \), and \( \sigma \) is injective.

Ex. \( R' \) an \( R \)-alg, \( M \) \( R' \)-module \( H = \text{Hom}_R(R', M) \).

\( \alpha: M \rightarrow H \) \( \alpha(m)(x) = xm \), \( \beta: H \rightarrow M \)

\( \beta(\theta) = \theta(1) \). Then \( \beta \) is a retraction of \( \alpha \).

So \( M \) is a direct summand of \( H \).

Lema. Consider comm. diagram with exact rows:

\[
\begin{array}{cccccc}
M' & \rightarrow & M & \rightarrow & M'' & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
N' & \rightarrow & N & \rightarrow & N'' & \rightarrow & 0 \\
0 & \rightarrow & N' & \rightarrow & N & \rightarrow & N'' \\
\end{array}
\]
This diagram defines the following exact sequence:

\[ \text{Ker} \gamma' \rightarrow \text{Ker} \gamma \rightarrow \text{Ker} \gamma'' \rightarrow \text{Coker} \gamma' \rightarrow \text{Coker} \gamma \rightarrow \text{Coker} \gamma'' \rightarrow \ldots \]

Moreover, if \( \alpha \) is injective then so is \( \gamma \), and if \( \beta ' \) is surjective then so is \( \gamma ' \).

**Pf.** Clearly \( \alpha \) restricts to \( \gamma \), since if \( \gamma'(v) = 0 \) then \( \gamma \alpha (v) = \alpha' \gamma'(v) = 0 \), so \( \alpha(v) \in \text{Ker} \gamma \).

Similarly, \( \beta \) restricts to \( \gamma \), \( \alpha \) restricts to \( \gamma ' \) and \( \beta ' \) to \( \gamma ' \). So it remains to define \( \delta \). To define \( \delta \), choose an \( m'' \in \text{Ker} \gamma '' \) through the diagram. Since \( \beta ' \) is surjective, \( m'' = \beta ''(m) \). We have \( \beta ' \gamma (m) = \gamma '' \beta (m) = 0 \). Since the lower seq. is exact, there is \( n' \) s.t. \( \alpha ' (n') = s(m) \), and its unique. Let \( \delta '(m'') \) be the image of \( n' \) in \( \text{Coker} \gamma ' \).

We need to check that \( \delta \) is well defined. To do so, choose another \( m'_1 \) s.t. \( \beta (m'_1) = m'' \). Let \( n'_1 \) be the unique elt s.t. \( \alpha (n'_1) = \delta (m'_1) \).

Since \( \beta (m - m'_1) = 0 \), there is \( m'' = m'_1 \) s.t. \( \delta (m') = m - m'_1 \). So \( \delta ' \gamma ' (m') = \gamma ' \delta (m') = \gamma ' (m - m'_1) = \delta ' (n' - n'_1) \). Hence \( \delta ' (m') = n' - n'_1 \) since \( \delta ' \) is injective. So \( n' \) and \( n'_1 \) have the same image in \( \text{Coker} \gamma ' \), and \( \delta \) is well defined.

Now we need to check that the sequence is exact.
We will only check exactness in \( \text{Ker} \gamma \). The other verifications are similar or easier.

Take \( m'' \in \text{Ker} \gamma \). As in the construction of \( \gamma \), take \( m \in M \) s.t. \( \beta(m) = m'' \) and \( n' \in N' \) s.t. \( \alpha'(n') = \gamma(m) \). Suppose \( m'' \in \text{Ker} \gamma \). Then \( n' = 0 \) in \( \text{Coker}(\gamma') \), so have \( m' \in M' \) s.t. \( \gamma(m') = n' \).

So \( \gamma(m') = \alpha'(\gamma(m')) = \alpha'(n') = \gamma(m) \). Hence \( \gamma(m - \alpha(m')) = 0 \) and \( m - \alpha(m') \in \text{Ker} \gamma \).

Since \( \beta(m - \alpha(m')) = \beta(m) = m'' \), we have

\[ m'' = \gamma(m - \alpha(m')) , \text{ so } m'' \in \text{Im} \gamma. \]

Conversely, suppose \( m'' \in \text{Im} \gamma \). Can assume \( m \in \text{Ker} \gamma \), and \( \gamma(m) = m'' \). So \( \gamma(m) = 0 \) and \( \alpha'(n') = 0 \). Hence \( n' = 0 \) since \( \alpha' \) is injective.

Thus \( \gamma(m'') = 0 \), so \( \text{Im} \gamma \subset \text{Ker} \gamma \). So \( \text{Im} \gamma = \text{Ker} \gamma \) and the sequence is exact in \( \text{Ker} \gamma \).

Thm. (Left exactness of \( \text{Hom} \)).

(1) \( 0 \to M' \to M'' \to 0 \) is exact iff \( \forall N \)

\[ 0 \to \text{Hom}(M'', N) \to \text{Hom}(M', N) \to \text{Hom}(M', N') \]

is exact.

(2) \( 0 \to N' \to N \to N'' \) is exact \( \Rightarrow \)

\[ \forall M \quad \text{Hom}(M, N') \to \text{Hom}(M, N) \to \text{Hom}(M, N'') \]

is exact.
**Proof.** Exactness of $M' \to M \to M'' \to 0$ means that $M'' = \text{Coker}(x)$. But the exactness of $0 \to \text{Hom}(M'', N) \to \text{Hom}(M', N) \to \text{Hom}(M, N)$ means that $\psi \in \text{Hom}(M, N)$ maps to $0$, i.e. $\psi x = 0$, if and only if $\exists \beta : M'' \to N$ s.t. $\beta \psi = \psi$. Thus this sequence is exact if and only if $M''$ has a UMP of \text{Coker}(x). So $M'' = \text{Coker}(x)$.

The proof of (2) is similar.

**Ex.** $0 \to M' \to M \to M'' \to 0$ exact \[\text{exact}\]

$0 \to \text{Hom}(M', N) \to \text{Hom}(M, N) \to \text{Hom}(M'', N) \to 0$

E.g. $0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/2 \to 0$

and take $\text{Hom}$ to $\mathbb{Z}/2$.

**Def.** A free presentation of $M$ is an exact sequence $0 \to F \to M \to 0$

where $F$ is free. If $F$ is free of finite rank, say this is a finite presentation.

**Prop.** If $M$ is an $R$-module, $M$ has $n$ generators. Then have a presentation exact seq.

$0 \to K \to R^g \to M \to 0$, and a present.

$R^g \to R^k \to M \to 0$.

$k = \text{Ker} \psi$, $\Sigma = \text{set of gens of } K$. 

**Def.** $P$ is projective if for any surj. map $\beta : M \to N$, every lin. map $\alpha : P \to N$ lifts to $x : P \to M$.

**Ex.** Free module is projective.

(can pick preimages).

**Thm.** TFAE:

1) $P$ is projective
2) Every short exact sequence $0 \to K \to M \to P \to 0$ splits
3) There is a module $K$ s.t. $K \oplus P$ free
4) Every exact sequence $N' \to N \to N''$ induces an exact sequence $\text{Hom}(P,N') \to \text{Hom}(P,N)$
5) Every surjective morphism $M \to N$ induces a surjection $\text{Hom}(P,M) \to \text{Hom}(P,N)$.

**Pf.** (1) $\Rightarrow$ (2). We have $M \to M/K = P \to N$ so lift $P \to M$.

(2) $\Rightarrow$ (3): $0 \to K \to RA \to P \to 0$

$RA = K \oplus P$

(3) $\Rightarrow$ (4) Obvious for free modules. But for projective it's a direct summand.

(4) $\Rightarrow$ (5) Special case: $M \to N \to 0$

(5) $\Rightarrow$ (1). Obvious.
Lemma (Exhauel) \[ \text{If } 0 \rightarrow L \rightarrow P \rightarrow M \rightarrow 0 \]
\[ 0 \rightarrow L' \rightarrow P' \rightarrow M' \rightarrow 0 \]
are exact, and \(P, P'\) projective, then we have an isom. of exact sequences
\[ 0 \rightarrow L \oplus P' \rightarrow P \oplus P' \rightarrow M \rightarrow 0 \]
\[ 0 \rightarrow L' \oplus P \rightarrow P' \oplus P' \rightarrow M' \rightarrow 0 \]

**Proof.** First establish an isom
\[ 0 \rightarrow L \oplus P' \rightarrow P \oplus P' \rightarrow M \rightarrow 0 \]
\[ \begin{array}{ccc}
\uparrow \lambda & \uparrow \theta & \uparrow \imath M \\
0 \rightarrow K \rightarrow P \oplus P' \rightarrow M \rightarrow 0 \\
\end{array} \]

where \(K = \text{Ker} \ (\alpha \beta)\)

To define \(\theta\), recall \(P\) is projective and \(\alpha\) is surjective. So \(\exists \pi: P' \rightarrow P\) s.t. \(P' \rightarrow P\)

let \(\theta = (1 \pi)\). Then \(\theta^{-1} = (1 - \pi)\)

And the right square is comm. Thus \(\theta\) induces \(\lambda: K \rightarrow L \oplus P'\).

Since \(L, L', P, P'\) play symmetric role, we get the desired statement.

**Prop.** If \(0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0\) is a short exact seq. and \(L\) fin. gen., \(M\) fin. pres. Then \(N\) is finitely presented.
\[ M = F / R, \ F \text{ f.g. free, } R \text{ f.gen.} \]

Now \[ N = M / \mathcal{L} \]. Let \( \tilde{L} \) be the preimage of \( \mathcal{L} \) in \( F \). If \( l_1, \ldots, l_n \) gen. of \( L \), their lifts \( \tilde{l}_1, \ldots, \tilde{l}_n \) together with generators \( z_1, \ldots, z_k \) of \( R \) are gen. of \( \tilde{L} + R \). But \[ F / \mathcal{L} + R = M / \mathcal{L} \].

Prop. \[ 0 \to L \to M \to N \to 0 \], \( L \) and \( N \) fin. presented \( \to \) \( M \) fin. presented.

First show \( M \) is fin. gen. Let \( \tilde{l}_1, \ldots, \tilde{l}_s \) generators of \( L \), \( \tilde{m}_1, \ldots, \tilde{m}_t \) gen. of \( N \). Let \( \tilde{m}_1, \ldots, \tilde{m}_t \) their lifts to \( M \). Then \( l_1, \ldots, l_s, m_1, \ldots, m_t \) generators of \( M \).

Now we have
\[ 0 \to RS \to RS + RT \to RT \to 0 \]
\[ \xymatrix{ \psi \ar[d] & B \ar[d] & \ar[d] \delta \ar[d] \\ 0 \to L \to M \to N \to 0 } \]

By Snake Lemma, we have an exact sequence
\[ 0 \to \ker \alpha \to \ker \beta \to \ker \delta \to 0. \]
We can choose $S$ and $T$ so that $\ker A$ and $\ker B$ are finitely generated. Hence by the above $\ker B$ is finitely generated, as desired.