Lecture 5. Exact sequences.

<u>Def.</u> $\dots \to M_{i-1} \xrightarrow{\alpha_{i-1}} M_i \xrightarrow{\alpha_i} M_{i+1} \to \dots$ sequence of module homom is exact at M_i if $\operatorname{Ker}\alpha_i = \operatorname{Im}\alpha_{i-1}$.
Exact = exact at all terms (except source and target).

<u>Ex.</u> $0 \to L \xrightarrow{\alpha} M$ is exact $\Leftrightarrow$ $L \xrightarrow{\alpha} M$ is injective.
$M \xrightarrow{\beta} N \to 0$ exact $\Leftrightarrow$ $M \xrightarrow{\beta} N$ surjective

$0 \to L \to M \to N$ exact $\Leftrightarrow$ $L = \operatorname{Ker}\beta$.
$L \to M \to N \to 0$ exact $\Leftrightarrow$ $N = \operatorname{Coker}\alpha$.
$0 \to L \xrightarrow{\alpha} M \xrightarrow{\beta} N \to 0$ exact $\Leftrightarrow$ $L = \operatorname{Ker}\beta$ and β is surjective $\Leftrightarrow$ $N = \operatorname{Coker}\alpha$ and α is injective.

If so, it's called a <u>short exact sequence</u>.
In this case $L \subset M$, $N = M/L$. So such sequences correspond to extensions M on N by L.

<u>Ex.</u> $0 \to L \to L \oplus N \to N \to 0$
<u>split</u> short exact sequence

<u>Prop.</u> $\forall \lambda \in \Lambda$, let $M'_\lambda \to M_\lambda \to M''_\lambda$. If each of these is exact, then

$$\bigoplus_\lambda M'_\lambda \to \bigoplus_\lambda M_\lambda \to \bigoplus_\lambda M''_\lambda, \quad \prod_\lambda M'_\lambda \to \prod_\lambda M_\lambda \to \prod_\lambda M''_\lambda$$

are exact, and vice versa.

<u>Pf.</u> Easy.

<u>Prop.</u> $0 \to M' \xrightarrow{\alpha} M \xrightarrow{\beta} M'' \to 0$ short exact sequence. $N \subset M$, $N' = \alpha^{-1}(N)$, $N'' = \beta(N)$.
Then $0 \to N' \to N \to N'' \to 0$ is short exact.

<u>Pf.</u> easy.

Def $\rho : M \to M'$ <u>retraction</u> of $\alpha : M' \to M$ if $\rho\alpha = id_{M'}$. Then α is injective and ρ surj. Also $\sigma : M'' \to M$ a <u>section</u> of $\beta : M \to M''$ if $\beta\sigma = 1_{M''}$. Then β is surjective and σ injective. $\boxed{M' \to M \to M'' \text{ splits if } M = M' \oplus M'' \text{ with obv. maps}}$

Prop. Let $M' \to M \to M''$ be a 3-term exact sequence. TFAE:

1) The sequence splits.

2) Have a retraction $M \to M'$ of α, and β is surj.

3) have a section $M'' \to M$ of β, and α is injective.

Pf. exercise.

Ex. R' an R-alg, M R'-module $H = Hom_R(R', M)$.

$\alpha : M \to H$ $\quad \alpha(m)(x) = xm$, $\rho : H \to M$

$\rho(\theta) = \theta(1)$. Then ρ is a retraction of α

So M is a direct summand of H.

Lemma. Consider comm. diagram with exact rows:

$$\begin{tikzcd}
 & M' \arrow[r, "\alpha"] \arrow[d, "\gamma'"] & M \arrow[r, "\beta"] \arrow[d, "\gamma"] & M'' \arrow[r] \arrow[d, "\gamma''"] & 0 \\
0 \arrow[r] & N' \arrow[r, "\alpha'"] & N \arrow[r, "\beta'"] & N'' &
\end{tikzcd}$$

This diagram defines the following exact sequence:

$$\operatorname{Ker}\gamma' \xrightarrow{\varphi} \operatorname{Ker}\gamma \xrightarrow{\psi} \operatorname{Ker}\gamma'' \xrightarrow{\partial} \operatorname{Coker}\gamma' \xrightarrow{\varphi'} \operatorname{Coker}\gamma \xrightarrow{\psi'} \operatorname{Coker}\gamma''.$$

Moreover, if α is injective then so is φ, and if β' is surjective then so is ψ'.

Pf. Clearly α restricts to φ, since if $\gamma'(v)=0$ then $\gamma\alpha(v) = \alpha'\gamma'(v)=0$, so $\alpha(v)\in\operatorname{Ker}\gamma$. Similarly, β restricts to ψ, α' restricts to φ' and β' to ψ'. So it remains to define ∂. To define ∂, chase an $m''\in\operatorname{Ker}\gamma''$ through the diagram. Since β is surjective, $m''=\beta(m)$, $m\in M$. We have $\beta'\gamma(m)=\gamma''\beta(m)=0$. Since the lower seq. is exact, there is n' s.t. $\alpha'(n')=\gamma(m)$, and it's unique. Let $\partial(m'')$ be the image of n' in $\operatorname{Coker}\gamma'$.

We need to check that ∂ is well defined. To do so, choose another m_1 s.t. $\beta(m_1)=m''$. Let n_1' be the unique elt s.t. $\alpha(n_1')=\gamma(m_1)$. Since $\beta(m-m_1)=0$, there is $m'\in M'$ s.t. $\alpha(m')=m-m_1$. So $\alpha'\gamma'(m')=\gamma\alpha(m')=\gamma(m-m_1)$ $=\alpha'(n'-n_1')$. Hence $\gamma'(m')=n'-n_1'$ since α' is injective. So n' and n_1' have the same image in $\operatorname{Coker}(\gamma')$, and ∂ is well defined.

Now we need to check that the sequence is exact.

We will only check exactness in $\operatorname{Ker}\gamma''$. The other verifications are similar or easier. Take $m'' \in \operatorname{Ker}\gamma''$. As in the construction of ∂, take $m \in M$ s.t. $\beta(m) = m''$, and $n' \in N'$ s.t. $\alpha'(n') = \gamma(m)$. Suppose $m'' \in \operatorname{Ker}\partial$. Then $n' \mapsto 0$ in $\operatorname{Coker}(\gamma')$, so have $m' \in M'$ s.t. $\gamma(m') = n'$. So $\gamma\alpha(m') = \alpha'\gamma'(m') = \alpha'(n') = \gamma(m)$. Hence $\gamma(m - \alpha(m')) = 0$ and $m - \alpha(m') \in \operatorname{Ker}\gamma$. Since $\beta(m - \alpha(m')) = \beta(m) = m''$, we have $m'' = \psi(m - \alpha m')$, so $m'' \in \operatorname{Im}\psi$.

Conversely, suppose $m'' \in \operatorname{Im}(\psi)$. Can assume $m \in \operatorname{Ker}\gamma$, and $\psi(m) = m''$. So $\gamma(m) = 0$ and $\alpha'(n') = 0$. Hence $n' = 0$ since α' is injective. Thus $\partial(m'') = 0$, so $\operatorname{Im}\psi \subset \operatorname{Ker}\partial$. So $\operatorname{Im}\psi = \operatorname{Ker}\partial$ and the sequence is exact in $\operatorname{Ker}\gamma''$.

<u>Thm.</u> (left exactness of Hom).

(1) $M' \to M \to M'' \to 0$ is exact iff $\forall N$

$$0 \to \operatorname{Hom}(M'', N) \to \operatorname{Hom}(M, N) \to \operatorname{Hom}(M', N)$$

is exact.

(2) $0 \to N' \to N \to N''$ is exact $\iff$

$\forall M$ $\operatorname{Hom}(M, N') \to \operatorname{Hom}(M, N) \to \operatorname{Hom}(M, N'')$ is exact.

Pf. Exactness of $M' \xrightarrow{\alpha} M \xrightarrow{\beta} M'' \to 0$ means that $M'' = \mathrm{Coker}(\alpha)$. But the exactness of

$0 \to \mathrm{Hom}(M'',N) \to \mathrm{Hom}(M,N) \to \mathrm{Hom}(M',N)$

means that $\varphi \in \mathrm{Hom}(M,N)$ maps to 0, i.e. $\varphi\alpha = 0$, if and only if $\exists! \gamma: M'' \to N$ s.t. $\gamma\beta = \varphi$. Thus this sequence is exact if and only if M'' has a UMP of $\mathrm{Coker}(\alpha)$. So $M'' = \mathrm{Coker}(\alpha)$.

The proof of (2) is similar.

Ex. $0 \to M' \to M \to M'' \to 0$ exact $\not\Leftarrow$

$0 \to \mathrm{Hom}(M',N) \to \mathrm{Hom}(M,N) \to \mathrm{Hom}(M'',N) \to 0$

E.g. $0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/2 \to 0$ and take Hom to $\mathbb{Z}/2$.

Def. A free presentation of a module M is an exact sequence $G \to F \to M \to 0$, where F, G are free. If F, G are of finite rk, say this is a finite presentation.

Prop. M R-module, M_λ generators. Then have an ~~presentation~~ exact seq.

$0 \to K \to R^{\oplus \Lambda} \xrightarrow{\varphi} M \to 0$, and a present.

$R\Sigma \to R\Lambda \to M \to 0$.

Pf. $K = \mathrm{Ker}\,\varphi$, Σ - set of gen. of K.

<u>Def.</u> P is projective if for any surj. map $\beta : M \twoheadrightarrow N$, every lin map $\alpha : P \to N$ lifts to $\gamma : P \to M$.

$$\begin{tikzcd} P \arrow[r, dashed, "\gamma"] \arrow[dr, "\alpha"'] & M \arrow[d, "\beta"] \\ & N \end{tikzcd}$$

<u>Ex.</u> Free module is projective. (can pick preimages).

<u>Thm.</u> TFAE:

1) P is projective

2) Every short exact sequence
$0 \to K \to M \to P \to 0$ splits

3) There is a module K s.t. $K \oplus P$ free

4) every exact sequence $N' \to N \to N''$ induces an exact sequence $\mathrm{Hom}(P,N') \to \mathrm{Hom}(P,N) \to \mathrm{Hom}(P,N'')$.

5) Every surjective morphism $M \twoheadrightarrow N$ induces a surjection $\mathrm{Hom}(P,M) \to \mathrm{Hom}(P,N)$.

<u>Pf.</u> (1) $\Rightarrow$ (2). We have $M \twoheadrightarrow M/_K = P = N$ so $\exists$ lift $P \to M$.

(2) $\Rightarrow$ (3) $0 \to K \to R\Lambda \to P \to 0$

$R\Lambda = K \oplus P$

(3) $\Rightarrow$ (4) Obvious for free modules. But for projective it's a direct summand.

(4) $\Rightarrow$ (5) Special case: $M \to N \to 0$

(5) $\Rightarrow$ (1). Obvious.

<u>Lemma</u> (Schanuel) If $0 \xrightarrow{i} L \xrightarrow{\alpha} P \to M \to 0$
$0 \xrightarrow{i'} L' \xrightarrow{\alpha'} P' \to M \to 0$
are exact, and P, P' projective, then we have an isom. of exact sequences

$$\begin{tikzcd} 0 \arrow[r] & L \oplus P' \arrow[r] \arrow[d] & P \oplus P' \arrow[r] \arrow[d] & M \arrow[r] \arrow[d, "1_M"] & 0 \\ 0 \arrow[r] & P \oplus L' \arrow[r] & P \oplus P' \arrow[r] & M \arrow[r] & 0 \end{tikzcd}$$

<u>Proof.</u> First establish an isom

$$\begin{tikzcd} 0 \arrow[r] & L \oplus P' \arrow[r] & P \oplus P' \arrow[r, "(\alpha\, 0)"] & M \arrow[r] & 0 \\ 0 \arrow[r] & K \arrow[r] \arrow[u, leftrightarrow, "\lambda"] & P \oplus P' \arrow[r, "(\alpha\, \alpha')"'] \arrow[u, leftrightarrow, "\theta"] & M \arrow[r] \arrow[u, leftrightarrow, "1_M"] & 0 \end{tikzcd}$$

where $K = \mathrm{Ker}(\alpha\, \alpha')$

To define θ, recall P is projective and α is surjective. So $\exists\, \pi : P' \to P$ s.t.

$$\begin{tikzcd} P' \arrow[d, "\pi"'] \arrow[dr] & \\ P \arrow[r] & M \end{tikzcd}$$

let $\theta = \begin{pmatrix} 1 & \pi \\ 0 & 1 \end{pmatrix}$. Then $\theta^{-1} = \begin{pmatrix} 1 & -\pi \\ 0 & 1 \end{pmatrix}$

And the right square is comm.
Thus θ induces $\lambda : K \xrightarrow{\sim} L \oplus P'$.
Since L, L', P, P' play symmetric role, we get the desired statement.

<u>Prop.</u> If $0 \to L \to M \to N \to 0$ is a short exact seq. and L fin. gen, M fin. pres. Then N is finitely presented.

$M = F/R$, F f.g. free, R f.gen.

Now $N = M/L$. Let $\tilde{L}$ be preimage of L in F. If $l_1, \dots, l_n$ gen. of L, Their lifts $\tilde{l}_1, \dots, \tilde{l}_n$ together with generators $r_1, \dots, r_k$ of R are gen. of $\tilde{L}+R$. But $F/\tilde{L}+R = M/L = N$.

<u>Prop.</u> $0 \to L \to M \to N \to 0$, L and N fin. presented $\Rightarrow$ M fin. presented.

<u>Pf.</u> First show M is fin. gen. Let $\{l_1, \dots, l_k\} = S$ generators of L, $\{n_1, \dots, n_s\} = T$ gen. of N. Let $\tilde{n}_1, \dots, \tilde{n}_s$ their lifts to M. Then $l_1, \dots, l_k, \tilde{n}_1, \dots, \tilde{n}_s$ generates M.

Now we have

$$\begin{array}{ccccccccc}
0 & \to & RS & \to & RS \oplus RT & \longrightarrow & RT & \to & 0 \\
 & & \downarrow \alpha & & \downarrow \beta & & \downarrow \gamma & & \\
0 & \to & L & \longrightarrow & M & \longrightarrow & N & \to & 0
\end{array}$$

By Snake lemma, we have an exact sequence

$0 \to \operatorname{Ker}\alpha \to \operatorname{Ker}\beta \to \operatorname{Ker}\gamma \to 0.$

We can choose S and T so that $\operatorname{Ker}\alpha$ and $\operatorname{Ker}\gamma$ are fin. generated. Hence by the above $\operatorname{Ker}\beta$ is f. gen, as desired.