Lecture 4. Modules

R a ring. **R-module**: abelian group M, written additively, with scalar mult. $R\times M\to M$ s.t.

$x(m_1+m_2)=xm_1+xm_2$

$x(ym)=(xy)m$

$1\cdot m=m.$

E.g. if R is a field, a module is a vector space. A $\mathbb{Z}$-module is an ab. group.

Submodule. $N\subset M$ ~~de~~ subgroup closed under scalar mult.

Ex. R is an R-module, submodules = ideals

$\mathfrak{a}\subset R$ ideal, M R-module $\Rightarrow \mathfrak{a}M\subset M$ submodule

Annihilator of $m\in M$ $\mathrm{Ann}(m)=\{x\in R \mid xm=0\}$.

$\mathrm{Ann}(M)=\{x\in R \mid \forall m\in M \;\; xm=0\}$. These are ideals.

Homomorphisms: M, N R-modules. $\varphi: M\to N$ homom. of groups.

Def. φ is a module homom if it commutes with scalar mult, i.e. $\varphi(\alpha m)=\alpha\varphi(m)$.

$\operatorname{Ker}\varphi\subset M$, $\operatorname{Im}\varphi\subset N$ submodules.

Isom = bijective homom. In this case φ^{-1} is also a homom.

Set of homomorphisms $\mathrm{Hom}_R(M,N)=\mathrm{Hom}(M,N)$.

This is also an R-module.

Endomorphisms. Homom $\varphi: M\to M$.

$\mathrm{Hom}(M,M)$ also denoted $\mathrm{End}_R(M)$ or $\mathrm{End}(M)$

E.g. $\forall x\in R$ $\mu_x: M\to M$ $\mu_x(m)=xm$. $\mu_x\in\mathrm{End}(M)$.

So have $\mu_R : R \to \mathrm{End}_R(M)$. $\subset \mathrm{End}_{\mathbb{Z}} M$. (note that the latter is noncommutative). Also $\mu : R \to \mathrm{End}_{\mathbb{Z}} M$ is the same thing as a module structure.

<u>Faithful module</u>: μ_R is injective, or $\mathrm{Ann}(M) = 0$.

E.g. R is a faithful module ($x \cdot 1 = 0 \Rightarrow x = 0$).

<u>Algebras</u>. $\varphi : R \to R'$, so R' is an R-algebra. Then any R'-module is also an R-module via restriction of scalars. In partic, R' is an R-module.

Suppose $R' = R/\mathfrak{a}$. Then an R-module M comes from an R'-module (descends to R') if $\mathfrak{a}M = 0$.

<u>Subalgebra</u>: $R'' \subset R'$ subring closed under scalar mult. by R. R'' generated by $x_1, \dots, x_n \in R'$ is the smallest subalg. containing them. We denoted it by $R[x_1, \dots, x_n]$. If $R'' = R'$, say R' is gen. by $x_1, \dots, x_n$. Note we can have relations between x_i.

<u>Fin. generated algebra over R</u>: $R = R[x_1, \dots, x_n]$.

<u>Residue modules</u>. (quotients). R ring, $M' \subset M$ modules. M/M' is an R-module.

$\kappa : M \to M/M'$.

$\varphi : M \to N$ homomorphism. Then φ descends to $\bar\varphi : M/M' \to N \Leftrightarrow \varphi|_{M'} = 0$.

$\bar\varphi : M/\mathrm{Ker}\,\varphi \hookrightarrow N$ injective $\quad \bar\varphi : M/\mathrm{Ker}\,\varphi \to \mathrm{Im}\,\varphi$ isom. first Noether isom

So M/M' has a UMP: $\mathrm{Hom}(M/M', N) = \{\varphi : M \to N \mid \varphi|_{M'} = 0\}$

Cyclic modules: M cyclic if $\exists m \in M$ $M = Rm$. Then $R/_{\mathrm{Ann}(M)} \cong M$. So cyclic modules ~~with fixed m~~ are exactly $R/\mathfrak{a}$, $\mathfrak{a} \subset R$ ideal.

Noether isomorphisms: $L \subset M \subset N$

$$\begin{array}{ccc} N & \to & N/_M \\ \downarrow & & \wr\!\wr\ \beta \\ N/_L & \to & N/_L \big/ M/_L \end{array}$$

second Noether isom.

$L, M \subset N$. $L + M = \{ l + m, l \in L, m \in M \}$.

$$\begin{array}{ccc} L & \longrightarrow & L/(L \cap M) \\ \downarrow & & \wr\!\wr\ \gamma \\ L+M & \longrightarrow & (L+M)/_M \end{array}$$

– third Noether isom.

Cokernels, coimages. R a ring $\alpha: M \to N$ linear map (homom).

$\mathrm{Coker}\,\alpha = N/_{\mathrm{Im}\,\alpha}$ $\quad \mathrm{Coim}(\alpha) = M/_{\mathrm{Ker}\,\alpha}$.

$\mathrm{Coim}(\alpha) \cong \mathrm{Im}(\alpha)$ by first Noether isom.

UMP of cokernel: $\mathrm{Hom}(\mathrm{Coker}\,\alpha, P) =$
$= \{ \varphi \in \mathrm{Hom}(N, P) \mid \varphi \circ \alpha = 0 \}$.

Free modules. Λ a set M an R-module, $m_\lambda \in M$, $\lambda \in \Lambda$. Say m_λ generate M if $M = \{ \sum_{\lambda \in \Lambda} a_\lambda m_\lambda \}$, $a_\lambda \in R$. almost all summands are 0. Similarly define submodule generated by m_λ (if it is not the whole M).

$\{m_\lambda\}$ linearly independent if $\sum a_\lambda m_\lambda = 0$ $\Rightarrow a_\lambda = 0$.

M <u>free</u> if it has a lin. indep. set of generators (a free basis). E.g. a vector space is always free.

M is <u>fin gen.</u> if it has a finite set of gen. We'll see below that any two bases have the same number of elements. So M <u>free of rank ℓ</u> if M has a basis of ℓ elements. Write $\mathrm{rank}(M) = \ell$.

<u>Ex.</u> $R^{\oplus\Lambda} = R\Lambda$ - restricted vectors

(x_λ), $x_\lambda \in R$, $x_\lambda = 0$ for almost all λ.

Has standard basis $e_\lambda = (0 \ldots \underset{\uparrow}{1} \cdot 0)$.

$\mathrm{Hom}(R\Lambda, M) = \mathrm{Maps}(\Lambda, M)$. (UMP of $R\Lambda$).

$\tilde{\psi} : R\Lambda \to M$ $\qquad \psi \qquad \psi(\lambda) = m_\lambda$

$\tilde{\psi}$ surjective $\Leftrightarrow$ m_λ gen. M
injective $\Leftrightarrow$ m_λ lin. indep.
bijective $\Leftrightarrow$ m_λ is a free basis.

M free of rk ℓ $\Leftrightarrow$ $M \simeq R^\ell$.

<u>Ex.</u> $\mathbb{Q}$ over $\mathbb{Z}$ is not free, not f-gen. as a module or even as an algebra

<u>Thm.</u> R a PID, E a free R-module, e_λ free basis, $F \subset E$ submodule, then F is free which has a basis indexed by a subset of Λ.

exists by Zermelo theorem

<u>Proof</u>. Well order Λ. For all λ, let $\pi_\lambda : E \to R$ be the λ-th projection. $\forall \mu$, let $E_\mu = \bigoplus_{\lambda \leq \mu} Re_\lambda$, $F_\mu = F \cap E_\mu$. Then $\pi_\mu(F_\mu) = \langle a_\mu \rangle \subset R$, as R is a PID. Choose $f_\mu \in F_\mu$ with $\pi_\mu(f_\mu) = a_\mu$. Set $\Lambda_0 = \{\mu \in \Lambda \mid a_\mu \neq 0\}$. We will show that $\{f_\mu \mid \mu \in \Lambda_0\}$ is a free basis of F.

First show that $\{f_\mu \mid \mu \in \Lambda_0\}$ are linearly independent. Suppose $\sum_{\mu \in \Lambda_0} c_\mu f_\mu = 0$ for some $c_\mu \in R$. Let $\Lambda_1 = \{\mu \in \Lambda_0, \ c_\mu \neq 0\}$ (finite set). Suppose $\Lambda_1 \neq \emptyset$. Let μ_1 be the greatest elem. of Λ_1. Then $\pi_{\mu_1}(f_\mu) = 0$ for $\mu < \mu_1$, as $f_\mu \in E_\mu$.

So $\pi_{\mu_1}\left(\sum c_\mu f_\mu\right) = c_{\mu_1} a_{\mu_1}$. But $c_{\mu_1} \neq 0$ and $a_{\mu_1} \neq 0$, a contradiction. So $\{f_\mu\}$ are lin. ind.

Now we prove the spanning property of f_μ.

Note that $F = \bigcup_{\lambda \in \Lambda_0} F_\lambda$. For $\lambda \in \Lambda_0$ let $\Lambda_\lambda = \{\mu \in \Lambda_0 \mid \mu \leq \lambda\}$. Suppose λ is least such that $\{f_\mu\}_{\mu \in \Lambda_\lambda}$ does not generate F_λ. This exists because our order is a well-order. Given $f \in F_\lambda$, let $f = \sum_{\mu \leq \lambda} c_\mu e_\mu$, $c_\mu \in R$, so $\pi_\lambda(f) = c_\lambda$.

But $\pi_\lambda(F_\lambda) = \langle a_\lambda \rangle$. So $c_\lambda = b_\lambda a_\lambda$ for some $b_\lambda \in R$.

Let $g = f - b_\lambda f_\lambda$. Then $g \in F_\lambda$, $\pi_\lambda(g) = 0$. So $g \in F_\nu$ for some $\nu \in \Lambda_0$ with $\nu < \lambda$.

Hence $g = \sum_{\mu \in \Lambda_\nu} b_\mu f_\mu$ for some $b_\mu \in R$.

So $f = \sum_{\mu \in \Lambda_\lambda} b_\mu f_\mu$, a contradiction.

Thus, $\{f_\mu\}$ is a basis, as desired.

<u>Ex.</u> $R = \mathbb{C}[x, y]$, $\langle x, y \rangle$ is not a free module.

<u>Direct product</u>: $\prod_{\lambda \in \Lambda} M_\lambda = (m_\lambda)$ (vectors)

<u>Direct sum</u>: $\bigoplus_{\lambda \in \Lambda} M_\lambda = (m_\lambda)$, almost all are 0. (restricted vectors).

$\bigcap$

$\prod_{\lambda \in \Lambda} M_\lambda$.

But if Λ is finite, they are the same.

$$Hom(L, \prod_\lambda M_\lambda) = \prod_\lambda Hom(L, M_\lambda)$$

$$Hom(\bigoplus_\lambda M_\lambda, L) = \prod_\lambda Hom(M_\lambda, L).$$