Lecture 4. Modules

Let $R$ be a ring. An $R$-module is an abelian group $M$, written additively, with scalar multiplication $R \times M \rightarrow M$ s.t.
\[ x(m_1 + m_2) = x m_1 + x m_2 \]
\[ x(y m) = (x y) m \]
\[ 1 \cdot m = m. \]

Example: if $R$ is a field, an $R$-module is a vector space.

A $Z$-module is an abelian group.

A submodule $N \subseteq M$ is a subgroup closed under scalar mult.

Example: $R$ is an $R$-module, submodules $= \text{ideals of } R$.

An ideal $M$, $R$-module $\Rightarrow \alpha M \subseteq M$ submodule.

Annihilator of $m \in M$ is $\text{Ann} \{x \in R \mid x m = 0\}.

\text{Ann} (M) = \{x \in R \mid \forall m \in M \quad x m = 0\}$. These are ideals.

Homomorphisms: $M, N$ $R$-modules, $\varphi : M \rightarrow N$

Homomorphisms are homomorphisms of groups.

Def. $\varphi$ is a module homomorphism if it commutes with scalar mult., i.e. $\varphi(x m) = x \varphi(m)$.

Hom $\in \text{End}_R(M)$, $\text{End}_R(M)$ is a module.

Isomorphism $\Rightarrow$ bijective homomorphism. In this case $\varphi^{-1}$ is also a homomorphism.

Set of homomorphisms $\text{Hom}_R (M, N) = \text{Hom}_R (M, N)$.

This is also an $R$-module.

Endomorphisms: $\text{End}_R (M) = \text{Hom}_R (M, M)$

Endomorphisms is an $R$-module.

Endomorphisms: $\text{End}_R (M)$ also denoted $\text{End}_R (M)$ or $\text{End}(M)$

E.g. $\forall x \in R \quad \mu_x : M \rightarrow M \mu_x(m) = x m. \quad \mu_x \in \text{End}_R (M)$.
So have \( \mu : R \rightarrow \text{End}_\mathbb{Z} M \). (note that the latter is noncommutative). Also \( \mu : R \rightarrow \text{End}_\mathbb{Z} M \) is the same thing as a module structure.

**Faithful module:** \( \mu \) is injective, or \( \text{Ann}(M) = 0 \).

E.g. \( R \) is a faithful module \((x \cdot 1 = 0 \Rightarrow x = 0)\).

**Algebras:** \( R^1 \rightarrow R \Rightarrow \) \( R^1 \) is an \( R \)-algebra.

Then any \( R \)-module is also an \( R^1 \)-module via restriction of scalars. In particular, \( R^1 \) is an \( R \)-module.

Suppose \( R^1 = R/\alpha \). Then an \( R \)-module \( M \) comes from an \( R^1 \)-module \( (\text{descends to } R^1) \) if \( \alpha M = 0 \).

**Subalgebra:** \( R'' \subset R^1 \) being closed under scalar mult. by \( R \). \( R'' \) generated by \( x_1, \ldots, x_n \in R^1 \) is the smallest subalgebra containing them. We denote it by \( R[[x_1, \ldots, x_n]] \). If \( R'' = R^1 \), say \( R^1 \) is gen. by \( x_1, \ldots, x_n \). Not we can have relations between \( x_i \).

**Fr. gen. algebra over \( R \):** \( R = R[x_1, \ldots, x_n] \).

**Residue module. (quotient).** \( R \) rig. \( M / M' \) modules. \( M / M' \) is an \( R \)-module.

\( \varphi : M \rightarrow N \) homomorphism. Then \( \varphi \) descends to \( \varphi : M / M' \rightarrow N \) \( \iff \varphi / M' = 0 \).

\( \bar{\varphi} : M / \ker \varphi \rightarrow N \) inj. \( \bar{\varphi} : M / \ker \varphi \rightarrow \text{Im} \bar{\varphi} \) isom. So \( M / M' \) has a UMP: \( \text{Hom}(M / M', N) \hookrightarrow \{ \bar{\varphi} : M \rightarrow N / \ker \varphi \}_{\text{isom}} \).
Cyclic modules: $M$ cyclic if $\text{Im} \phi = M$

$M = R \cdot m$. Then $R/\text{Ann}(M) \cong M$. So cyclic modules with fixed $M$ are exactly $R/\mathfrak{a}$, or $\mathfrak{a}$ a $R$ ideal.

Noether isomorphisms: $\text{LCM}CN$

$N \cong N/M$  
$\downarrow$  
$N/\mathfrak{a} \cong (N/\mathfrak{a})/M/\mathfrak{a}$

$L, M \in N, L + M = \{l + m, l \in L, m \in M\}$.

$L \rightarrow L/(L \cdot 0)M$

$\downarrow$

$L + M \rightarrow (L + M)/M$

Cokernels, coimages: $R$ a ring $\phi: M \rightarrow N$ linear map (homom).

$\text{Coker} \phi = N/\text{Im} \phi$  
$\text{Coim} \phi = M/\ker \phi$.

$\text{Coim} \phi \cong \text{Im} \phi$. By first Noether isom.

UMP of cokernel: $\text{Hom} (\text{coker} \phi, P) = \{ \psi \in \text{Hom} (N, P) | \psi \circ \phi = 0 \}$.

Free modules: A set $M$ an $R$-module,

$m, n \in M$. Say $M$ generate $M$ if $M = \{ \sum_{\alpha \in \Lambda} a_{\alpha} m_{\alpha} | a_{\alpha} \in R \}$.

Almost all monomials are 0. Similarly define submodule generated by $m_{\alpha}$ (if it is not the whole $M$).

$\{m_{\alpha}\}$ linearly independent if $\sum a_{\alpha} m_{\alpha} = 0$  
$\Rightarrow a_{\alpha} = 0$. 
M free if it has a lin. indep. set of
generators (a basis). E.g. a vector space
is always free.

M is fin. gen. if it has a finite set of gen.
We'll see below that any two linear
have the same number of elements.
So M free of rank $\ell$ if $M$ has a basis of
$\ell$ elements. Write rank $(M) = \ell$.

Ex. $R^\infty = RA$ - restricted vectors
$(x_\ell) \ : \ x_\ell \in R \ \ , \ \ x_\ell = 0 \ \text{for almost all } \ell$.
Has standard basis $e = (0, 1, 0)$.

$\text{Hom}(RA, M) = \text{Maps}(\Lambda, M)$. (U^M of RA).

$\Psi : RA \to M$

$\Psi$ surjective $\Leftrightarrow M$, gen. $M$

injective $\Leftrightarrow M$, lin. indep.

bijective $\Leftrightarrow M^\times$ is a free basis.

$M$ free of $\ell$ $\Leftrightarrow M \cong \mathbb{R}^\ell$.

Ex. $Q$ over $\mathbb{Z}$ is not free, not f.-gen., as
a module or even as an algebra.

Thm. $R$ a PIB, $E$ a free $R$-module,
$\ell$ a free basis, $FCE$ submodule, then $F$ has
which has a basis indexed by a subset
of $\Lambda$. 
Proof. Well order \( \Lambda \). For all \( \lambda \), let \( \pi_\lambda: E \to \mathbb{R}^\lambda \) be the \( \lambda \)-th projection. For \( \mu \), let \( E'_\mu = \bigoplus_{\ell \neq \mu} R_{e_\ell} \), \( F_\mu = F \cap E'_\mu \). Then \( \pi_\mu(F_\mu) = \langle a_\mu \rangle \subset \mathbb{R} \), as \( \mathbb{R} \) is a P.D. Choose \( f_\mu \in F_\mu \) with \( \pi_\mu(F_\mu) = a_\mu \). Set \( \Lambda_0 = \{ \mu \mid \mu \in \Lambda, a_\mu \neq 0 \} \). We will show that \( \{ f_\mu \mid \mu \in \Lambda_0 \} \) is a free basis of \( F \).

First show that \( \{ f_\mu \mid \mu \in \Lambda_0 \} \) are linearly independent. Suppose \( \sum_{\mu \in \Lambda_0} c_\mu f_\mu = 0 \) for some \( c_\mu \in \mathbb{R} \). Let \( \Lambda_1 = \{ \mu \in \Lambda_0 : c_\mu 
eq 0 \} \) (finite set). Suppose \( \Lambda_1 \neq \emptyset \). Let \( \mu_1 \) be the greatest elem. of \( \Lambda_1 \). Then \( \pi_{\mu_1}(f_\mu) = 0 \) for \( \mu < \mu_1 \), as \( f_\mu \in E'_\mu \).

So \( \pi_{\mu_1} \left( \sum_{\mu \in \Lambda_0} c_\mu f_\mu \right) = c_{\mu_1} a_{\mu_1} \). But \( c_{\mu_1} \neq 0 \) and \( a_{\mu_1} \neq 0 \), a contradiction. So \( \{ f_\mu \mid \mu \in \Lambda_0 \} \) are lin. ind.

Now we prove the spanning property of \( f_\mu \). Let \( \Lambda_1 = \{ \mu \in \Lambda_0 : \mu \leq \lambda \} \). Suppose \( \Lambda_2 \) is least such that \( \{ f_\mu \mid \mu \in \Lambda_2 \} \) does not generate \( F_\lambda \). This exists because our order is a well-order. Given \( f \in F_\lambda \), let \( f = \sum_{\mu \in \Lambda_1} c_\mu f_\mu \), \( c_\mu \in \mathbb{R} \), so \( \pi_{\lambda}(f) = (c_{\lambda}) \).

But \( \pi_{\lambda}(F_\lambda) = \langle a_{\lambda} \rangle \). So \( c_{\lambda} b_\lambda a_{\lambda} \) for some \( b_\lambda \in \mathbb{R} \).

Let \( g = f - b_\lambda f_\lambda \). Then \( g \in F_\lambda \), \( \pi_{\lambda}(g) = 0 \). So \( g \in F_\nu \) for some \( \nu \in \Lambda_0 \) with \( \nu < \lambda \).
Hence \( g = \sum_{\mu \in \Lambda} b_{\mu} f_{\mu} \) for some \( b_{\mu} \in R \).

So \( f = \sum_{\mu \in \Lambda} b_{\mu} f_{\mu} \), a contradiction.

Thus, \( \{ f_{\mu} \} \) is a basis, as desired.

Ex. \( R = \mathbb{C}[x, y] \), \( \langle x, y \rangle \) is not a free module.

**Direct product:** \( \prod_{\lambda \in \Lambda} M_{\lambda} = (m_{\lambda}) \) (vectors)

**Direct sum:** \( \bigoplus_{\lambda \in \Lambda} M_{\lambda} = (m_{\lambda}) \), almost all \( \lambda \in \Lambda \) are 0. (restricted vectors).

But if \( \Lambda \) is finite, they are the same.

\[
\text{Hom}(L, \prod_{\lambda} M_{\lambda}) = \prod_{\lambda} \text{Hom}(L, M_{\lambda})
\]

\[
\text{Hom}(\bigoplus_{\lambda} M_{\lambda}, L) = \prod_{\lambda} \text{Hom}(M_{\lambda}, L).
\]