Lecture 22. -1-

Completions.

Let R be a ring, M a module with a filtr. $F^\circ M$. Then M has a topol. Open sets are arbitrary unions of $m + F^n M$ $\forall m, n$. This is a top since if

$m + F^n M \cap m' + F^{n'} M$ ~~...~~ $\neq \emptyset$

and $n \geq n'$ then have

x s.t. $x - m \in F^n M$, $x - m' \in F^{n'} M \subset F^{n} M$

$\Rightarrow m - m' \in F^{n} M$ so $m' + F^{n'} M \subset m + F^{n} M$

so int is $m + F^n M$.

additi. $M \times M \to M$ cont.

$M \to M$, $x \to m + x$ is a homeom.

$R \supset \mathfrak{a}$, give R $\mathfrak{a}$-adic filtr.

If filt. of M is an $\mathfrak{a}$-filt. then action is continuous.

Further, if filtr. is stable, then it yields the same top as $\mathfrak{a}$-adic filtr. as

$F^n M \supset \mathfrak{a}^n M \supset \mathfrak{a}^n F^{n'} M = F^{n+n'} M$. for some n' and any n.

Will use $\mathfrak{a}$-adic top. and rad(R) if R semilocal ~~for~~ unless specified otherwise.

$N \subset M$ submod. Closure $\overline{N} = \bigcap_n N + F^n M$

So each $F^n M$ closed, and $\{0\}$ closed $\Leftrightarrow$

$\bigcap F^n M = 0$.

M is separated (Hausdorff) $\Leftrightarrow$ 0 closed.

M is discrete ($\forall \{m\}$ both open and closed) $\Leftrightarrow$ $\{0\}$ is open.

<u>Cauchy sequences:</u> m_n Cauchy if $\forall n_0\ \exists n_1$ s.t.

$m_n - m_{n'} \in F^{n_0} M \quad \forall n, n' \geq n_1$.

$\Leftrightarrow$ $m_n - m_{n+1} \in F^{n_0} M \quad \forall n \geq n_1$.

An $m \in M$ is a limit of m_n if $\forall n_0$ $\exists n_1$ s.t. $m_n - m \in F^{n_0} M \quad \forall n \geq n_1$.

If every Cauchy sequence has a limit, call M complete (limit does not have to be unique).

Cauchy sequences for M forms a module. Sequences with 0 as a limit form a submod. The quotient is denoted

by $\hat{M}$ and is called the separated completion. Have a can. homom. $\chi : M \to \hat{M}$. If M is complete but not sep. then χ is surj but not inj. χ univ. cont. lin. map into sep. and complete filtered $\hat{R}$-module.
Then $\hat{R}$ is a ring, $\chi : R \to \hat{R}$ ring hom, $\hat{M}$ an $\hat{R}$-mod. Also $M \to \hat{M}$, $R\text{-mod} \to \hat{R}\text{-mod}$ is a linear functor.

E.g. $\widehat{R[X_1, \dots, X_n]} = R[[X_1, \dots, X_n]]$.

$\mathbb{Z} \supset (p)$, $\mathbb{Z} \supset p\mathbb{Z} \supset p^2\mathbb{Z} \supset \cdots$

$\hat{\mathbb{Z}} = \mathbb{Z}_p$ p-adic int.

<u>Prop.</u> $R \supset \mathfrak{a}$. Then $\hat{\mathfrak{a}} \subset \mathrm{rad}(\hat{R})$.

<u>Pf.</u> $\hat{R}$ is complete in $\hat{\mathfrak{a}}$-adic top.
So for any $x \in \hat{\mathfrak{a}}$, $\frac{1}{1-x} = 1 + x + x^2 + \cdots$
So $\hat{\mathfrak{a}} \subset \mathrm{rad}(\hat{R})$ by (3.2).

<u>Ex.</u> $1 + 2 + 4 + 8 + \cdots$ in 2-adics is -1.

<u>Inverse limits:</u> R a ring, Q_n R-mod, $\alpha_n^{n+1} : Q_{n+1} \to Q_n$, can define $\varprojlim Q_n$ the submodule

of $\prod Q_n$ of (q_n) with $\alpha_n^{n+1} q_{n+1} = q_n$ for all n.

~~Given Q_n and α_n^{n+1} $\forall n \in \mathbb{Z}$,~~

Let $\theta : \prod Q_n \to \prod Q_n$

$\theta(q_n) = q_n - \alpha_n^{n+1} q_{n+1}$.

Then $\varprojlim Q_n = \operatorname{Ker}\theta$.

$\varprojlim Q_n$ has UMP: Given $\beta_n : P \to Q_n$,

$\alpha_n^{n+1}\beta_{n+1} = \beta_n$, $\exists!\, \beta : P \to \varprojlim Q_n$ $\pi_n \beta = \beta_n$ $\forall n$.

Also $\varprojlim \operatorname{Hom}(P, Q_n) = \operatorname{Hom}(P, \varprojlim Q_n)$.

(inverse limit is dual to direct limit). $R = \mathbb{C}[x_1, \ldots, x_N]$

Ex. $\varprojlim R/\langle x_1, \ldots, x_n\rangle^N = \mathbb{C}[[x_1, \ldots, x_n]]$

$\varprojlim \mathbb{Z}/p^n\mathbb{Z} = \mathbb{Z}_p$.

L. For $n \geq 1$ cons. comm. diag with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & Q'_{n+1} \arrow[r, "\gamma'_{n+1}"] \arrow[d, "\alpha'^{\,n+1}_n"] & Q_{n+1} \arrow[r, "\gamma_{n+1}"] \arrow[d, "\alpha^{n+1}_n"] & Q''_{n+1} \arrow[r] \arrow[d, "\alpha''^{\,n+1}_n"] & 0 \\
0 \arrow[r] & Q'_n \arrow[r, "\gamma'_n"] & Q_n \arrow[r, "\gamma_n"] & Q''_n \arrow[r] & 0
\end{tikzcd}$$

Then the induced sequence

$$0 \to \varprojlim Q'_n \to \varprojlim Q_n \to \varprojlim Q''_n$$

is exact. Moreover, the last map is exact if the Q'_n satisfy the Mittag-Leffler cond:

$\forall n \quad Q_n \supset \alpha_n^{n+1} Q_{n+1} \supset \cdots \supset \alpha_n^m Q_m \supset \cdots$ stabilize.

i.e. $\alpha_n^m Q_m = \alpha_n^{m+k} Q_{m+k} \quad \forall k \geq 0.$

Pf. We have a comm. diagram

$$\begin{tikzcd}
0 \arrow[r] & \prod Q'_n \arrow[r, "\prod \gamma'_n"] \arrow[d, "\theta"'] & \prod Q_n \arrow[r, "\prod \gamma_n"] \arrow[d, "\theta'"] & \prod Q''_n \arrow[r] \arrow[d, "\theta''"] & 0 \\
0 \arrow[r] & \prod Q'_n \arrow[r] & \prod Q_n \arrow[r] & \prod Q''_n \arrow[r] & 0
\end{tikzcd}$$

with exact rows, where θ is as above. Snake lemma yields the exact sequence as above and an injection

Coker $\hat{\gamma} \hookrightarrow$ Coker θ'. Assume Q_n' satisfy the ML cond. Then Coker $\theta' = 0$ by exer. above. So Coker $\hat{\gamma} = 0$. □.

Prop. R a ring, M a module, $F^\circ M$ filtr. Then $\hat{M} = \varprojlim M/F^n M$.

Pf. Define $\alpha: \hat{M} \to \varprojlim M/F^n M$. Given a Cauchy seq. m_ν, let q_n be the residue of m_ν $\overbrace{\text{in } M/F^n M}$ for $\nu \gg 0$. Then q_n is indep of ν, because seq. is Cauchy. Clearly, q_n is the residue of q_{n+1} in $M/F^n M$. Also m_ν has 0 as a limit $\iff$ $q_n = 0$ $\forall n$. Let α be defined by $\alpha(m_\nu) = q_n$. It is easy to check that it is well defined, linear, and injective.

As to surjectivity, given $q_n \in \varprojlim M/F^n M$, for each ν, lift $q_\nu \in M/F^\nu M$ to $m_\nu \in M$. Then $m_\mu - m_\nu \in F^\nu M$ for $\mu \geq \nu$. Thus m_ν is Cauchy, so α is surj; hence an isom

Ex. R ring, M a mod, $F^{\cdot}M$ filt.

$$\begin{array}{ccccccccc} 0 & \to & F^{n+1}M & \to & M & \to & M/F^{n+1}M & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & F^{n}M & \to & M & \to & M/F^{n}M & \to & 0 \end{array}$$

We have

$$0 \to \varprojlim F^n M \to M \xrightarrow{\varkappa} \hat{M}$$

But $\varkappa$ is not surj if M is not complete.

So $\varprojlim$ not always exact, Coker θ not always 0.

Prop. A local ring, m max ideal. Then $\hat{A}$ is a local ring with max ideal $\hat{m}$.

Pf. $\hat{A}/\hat{m} = A/m$ $\quad (\hat{A} = \varprojlim A/m^k)$, so $\hat{m}$ is max.

Also any $z \notin \hat{m}$ is invertible. ~~Indeed~~,

~~let us look for z^{-1}~~ in the form

$z' \cancel{= \sum_{n=0}^{\infty} z_n , \quad z_n \in m^n}$

We have ~~to take~~ $z_0 = \bar{z}^{-1}$ ~~where $\bar{z}$ is~~ image ~~of~~ z in A/m.

Namely, we will construct z^{-1} as $\lim_{n\to\infty} w_n$, where $zw_n - 1 \in m^{n+1}$.
Let w_0 be any lift of $\bar{z}^{-1}$, where $\bar{z} \in k = A/m$ is the ~~image~~ image of z.
If w_{n-1} is constructed,
let $w_n = w_{n-1} + x$, $x \in m^n$.
So we need $zx + zw_{n-1} - 1 \in m^{n+1} \iff$
$\bar{z}\,\bar{x} + \overline{(zw_{n-1}-1)} = 0$ in m^n/m^{n+1}.
So we set $\bar{x} = -\bar{z}^{-1}\overline{(zw_{n-1}-1)}$ and pick x to be any lift of $\bar{x}$. $\square$.

$R \supset \mathfrak{a}$ $\varkappa : R \to \hat{R}$ can. map. $\forall t \in R$, $t_n \in R/\mathfrak{a}^n$ residue.

<u>L.</u> $\varkappa(t)$ is a unit $\iff$ t_n is a unit for any n.

<u>Pf.</u> in the notes.

Let $T = \varkappa^{-1}(\hat{R}^\times)$

Then $T = \{t \in R \mid t$ lies in no max ideal containing $\mathfrak{a}\}$. (i.e. $t \neq 0$ mod any max ideal cont $\mathfrak{a}$).

Let $S = 1+\mathfrak{a}$. Then $S \subset T$ (as no max ideal can contain both x and $1+x$).

So we get

$$\begin{tikzcd}
R \arrow[d] \arrow[rr, bend left] & & \hat{R} \\
S^{-1}R \arrow[r, "\sigma"] & T^{-1}R \arrow[r, "\tau"] & \hat{R}
\end{tikzcd}$$

also $R/\mathfrak{a}^n = S^{-1}R/\mathfrak{a}^n S^{-1}R = T^{-1}R/\mathfrak{a}^n T^{-1}R$.

So $\hat{R} = \widehat{S^{-1}R} = \widehat{T^{-1}R}$.

E.g. if m max ideal then $\hat{R} = \hat{R}_m$.

Finally, assume R is Noetherian.

Then σ, τ are inj. Indeed, supp.

$\tau\sigma(x/s) = 0$. Then $\chi(x) = 0$ as $\chi(s)$ is a unit. So $x \in \bigcap \mathfrak{a}^n$ By Krull int. thm, $\exists\, s' \in S$ s.t. $s'x = 0$. So $\frac{x}{s} = 0$ in $S^{-1}R$.

Thus σ, τ are inj.

<u>Thm.</u> (Exactness of ~~local~~ completion).

R Noetherian, $\mathfrak{a}$ an ideal. Then for f-gen modules M, the functor $M \to \hat{M}$ is exact.

Pf. Let $0 \to M' \to M \to M'' \to 0$ be an exact seq. of f.g. modules. So $F^nM' = M' \cap \mathfrak{a}^n M$. By Artin–Rees lemma F^nM' form an $\mathfrak{a}$-stable filtr. So it yields the same top as adic compl. Also Mittag-Leffler cond is satisfied as α_n^{n+1} are surj. So statement foll. from prev. prop.

Cor. R Noeth., M f.g. $\mathfrak{a}$ ideal. Then $\hat{M} \cong \hat{R} \otimes_R M$.

Pf. Since f-r is exact, this follows from the Watts thm.

~~Cor. $\hat{R}$ is a flat R-module (in the prev. sett.)~~

~~Cor. R Noeth. $\mathfrak{a}, \mathfrak{b} \subset R$ M f.g. module~~

Cor. R Noeth, $\mathfrak{a}, \mathfrak{b} \subset R$ ideals, M f.g. mod. Then (1) $\widehat{\mathfrak{b}M} = \mathfrak{b}\hat{M} = \hat{\mathfrak{b}}\hat{M}$ and

(2) $\widehat{\mathfrak{b}^n} = \mathfrak{b}^n\hat{R} = \mathfrak{b}^n\hat{R} = (\mathfrak{b}\hat{R})^n = \hat{\mathfrak{b}}^n$ for $n \geq 0$.

Pf. The inclusion $\mathfrak{b}M \hookrightarrow M$ induces a comm. square.

$$\begin{tikzcd} \hat{R} \otimes \mathfrak{b}M \arrow[r] \arrow[d] & \hat{R} \otimes M \arrow[d] \\ \widehat{\mathfrak{b}M} \arrow[r] & \hat{M} \end{tikzcd}$$

The image of the top map is $\mathfrak{b}(\hat{R} \otimes M)$.

The two vert. arrows are isom. and the bottom map is inj. by exactness of completion. So $(\mathfrak{b}M)^\wedge = \mathfrak{b}\hat{M}$.

Taking $R = M$ yields $\hat{\mathfrak{b}} = \mathfrak{b}\hat{R}$. So $\mathfrak{b}\hat{M} = \mathfrak{b}\hat{R}\hat{M} = \hat{\mathfrak{b}}\hat{M}$. This gives (1).

In (1) taking $\mathfrak{b}^m$ for $\mathfrak{b}$, R for M yields $(\mathfrak{b}^n)^\wedge = \mathfrak{b}^n\hat{R}$. So $\hat{\mathfrak{b}} = \mathfrak{b}\hat{R}$. So $(\mathfrak{b}\hat{R})^n = (\hat{\mathfrak{b}})^n$.

But $\mathfrak{b}^n R' = (\mathfrak{b}R')^n$ for any R-alg R'.

Thus (2) holds

<u>Cor.</u> R Noeth, $\mathfrak{a}$ an ideal. Then $\hat{R}$ is flat.

<u>Pf.</u> Let $\mathfrak{b}$ be any ideal. Then $\hat{R} \otimes \mathfrak{b} = \hat{\mathfrak{b}}$ and $\hat{\mathfrak{b}} = \mathfrak{b}\hat{R}$ by the above.

So $\hat{R}$ is flat by the ideal criterion of flatness.

<u>Ex.</u> $\hat{R} \otimes M \neq \hat{M}$ for general M.

Ex: $R = \mathbb{C}[t]$, $\mathfrak{a} = (t)$, $M = \mathbb{C}[t] \otimes \underset{\substack{\uparrow \\ \text{vect. space}}}{V}$

L. R a ring, $\alpha: M \to N$ map of modules, $F^\circ M$, $F^\circ N$ filtr. Assume $\alpha F^n M \subset F^n N$ for all n. Ass $F^n M = M$, $F^n N = N$ for $n \ll 0$. If the induced map $G^\circ \alpha$ is inj or surj then so is $\hat{\alpha}$.

Pf. In the notes.

L. R a ring, $\mathfrak{a}$ an ideal, M a mod, $F^\circ M$ an $\mathfrak{a}$-filtr. Ass. R is complete, M separated, $F^n M = M$, $n \ll 0$.
Ass. $G^\circ M$ is module finite over $G^\circ R$. Then M is complete and module finite over R.

Pf. Notes.

Prop. R a ring, $\mathfrak{a} \subset R$, M a mod. Ass. R complete, M separated. Ass $G^\circ M$ Noeth as $G^\circ R$ mod. then M is Noeth, and any $N \subset M$ is complete.

Pf notes.

Thm. R a ring, $\mathfrak{a}$ ideal.
if R is Noetherian, so is $\hat{R}$.

Pf. $G^\circ R$ is alg. fin. over $R/\mathfrak{a}$
so Noetherian by the Hilb. basis thm.
But $G^\circ R = G^\circ \hat{R}$. So $\hat{R}$ is Noetherian
by the above lemma. $\square$

Ex. k Noeth, $R = k[[X_1, \dots, X_n]]$ compl. of $k[X_1, \dots, X_n] \Rightarrow$ Noeth.
k dom $\Rightarrow R$ dom.
k ~~local~~ a field $\Rightarrow R$ local.

Thm. (UMP for power ser) R' an R-alg
~~Hom$(R[[X_1, \dots, X_n]], R') =$~~
$\mathfrak{b} \subset R'$
R' sep & compl in $\mathfrak{b}$-adic top
Then $\exists!$ map $R[[X_1, \dots, X_n]] \to R'$ $x_i \to \bar{x}_i \in \mathfrak{b}$

Thm. A complete Noeth. local alg $k = A/\mathfrak{m}$. Then $A = k[[X_1, \dots, X_n]]/\mathfrak{a}$. If A regular, then choose $\mathfrak{a} = 0$.

Pf. Pick gen. $x_1, \dots, x_r$ of $\mathfrak{a}$. Then get surj. $k[[X_1, \dots, X_r]] \to R$. If choose them minimally in the reg. case, $\mathfrak{a} = 0$