The dimension of $M$
\[ \dim M = \max \{ \dim (R/I) : I \text{ a chain of primes} \} \]
Assume $R$ Noetherian, $M$ f.gen. Then $M$ has fin. many min. assoc. primes. They are also min. primes in $\text{Supp} M$. Thus
\[ \dim M = \max \{ \dim R/I_0 : I_0 \in \text{Supp} M \text{ minimal} \} \]

Parameter. $R$ a ring, $M \cong R$-mod.
Denote $\mathfrak{m}$ of max. ideals in $\text{Supp} M$ as $\text{rad} M$, and call it the radical of $M$.
If there are fin. many such max. ideals, call $M$ semilocal.
Call $q < \text{rad} M$ a parameter ideal for $M$ if $M/qM$ is Artinian.
\[ \text{Ex: } M = R = \left[ E[x_1, \ldots, x_e] / I \right]_m \text{ then } \text{rad } M = m \]
Assume $M$ is f.gen. Then $\text{Supp } M = V(\text{Ann } M)$
So $M$ is semilocal $\iff R/\text{Ann } M$ is a semilocal ring.

Assume $R$ Noetherian, so $M$ Noetherian.
Fix $q < R$. Then $M/qM$ is Artinian $\iff \ell(M/qM) < \infty$.
But $\ell(M/qM) < \infty$ $\iff \text{Supp } (M/qM)$ consists of fin. many max. ideals, etc.
Also
$$\text{supp} \left( \frac{M}{qM} \right) = \text{Supp}M \setminus V(q) = V(\text{Ann} M) \setminus V(q)$$

$$= V(\text{Ann} M + q).$$

Set $q' = \text{Ann} M + q$.

Then $M/qM$ is Artinian $\implies V(q')$ consists of f.m. max ideals

$$\implies (\text{Akhizuki-Hopkins}) \ R/q', \text{ Artinian}.$$  
So by exer 19.18 $R/q'$, Artinian

$$\implies q'$ is cont a product of max ideals each containing $q'$.

So each lies in $\text{supp} M$, hence contains $\text{rad}(M)$. Thus $R/q'$ Artinian $\iff q' \supset m^n$ for some $n$. Assume $M$ semilocal, so $\text{Supp} M$ has f.m. many max ideals. Then their product is cont in $m$. Thus, conversely, if $q' \supset m^n$ for some $n > 0$, then $R/q'$ Artinian. Thus:

Prove: $q$ is a parau ideal $\iff$

$m \supset q' \supset m^n$ for some $n$, or

iff $m = \sqrt{q'} (\iff V(m) = V(q'))$.

In particular, $m^n$ is a parau ideal $\forall n$. 
Assume \( q \) is a parameter ideal. Then the Hibi-Keel-Samuel polynomial \( P_q(M, n) \) exists. Also \( P_{m}(M, n) \) exists, and the two polynomials have the same degree. (by exercise above), since \( m = \sqrt{q} \) and \( P_{m} = P_{q} \). Thus the degree is the same for any parameter ideal. Denote this common degree by \( d(M) \).

(See ex. R local ring of a variety, \( M \) f.g. \( R \)-module).

Alternatively, \( d(M) = \text{order of polyg at } 1 \) of \( H(E_{M}, t) \) (it is 1 less than that of \( E_{M}, t+1 \)).

Let \( s(M) \) be the smallest \( s \) such that \( x_1 \ldots, x_s \in M \) with \( l(M/x_1 \ldots, x_s M) < \infty \). (if \( l(M) < \infty \) then \( s(M) = 0 \)). We say that \( x_1 \ldots, x_s \in M \) form a system of parameters for \( M \) if \( s = s(M) \) and \( l(M/x_1 \ldots, x_s M) \) holds. Such a sop generates a parameter ideal.
Lemma.  R Noeth, $M \neq 0$ a Noeth semilocal module, $Q$ a prun ideal of $M$, $x \in \text{rad } M$.

Let $K = \text{Ker } (M \rightarrow M)$.

1. $s(M) \leq s(M/xM) + 1$.

2. $\dim (M/xM) \leq \dim M - 1$ if $x \neq 0$ for any $\emptyset \neq \text{supp } M$ with $\dim R/\emptyset = \dim M$.

3. $\deg (p_q (K, n) - p_q (M/xM, n)) \leq d(M) - 1$.

Proof. For (1), let $x \in s(M/xM)$. Then $x_1, \ldots, x_s \in \text{rad } (M/xM)$ with $\ell (M/\langle x, x_1, \ldots, x_s \rangle M) < \infty$.

Also $\text{supp } M/xM = \text{supp } M/\text{N} (\langle x \rangle)$. But $x \in \text{rad } M$, so $\text{supp } M/xM$ and $\text{supp } M$ have the same max. ideals. So $\text{rad } (M/xM) = \text{rad } M$.

Thus $s(M) \leq s + 1 \implies (1)$.

For (2) Take chain $\emptyset_0 \supsetneq \emptyset \supsetneq \emptyset \supsetneq \emptyset_0$ in $\text{supp } M/xM$.

Again $\text{supp } M/xM = \text{supp } M/\text{N} (\langle x \rangle)$. So $x \in \emptyset_0 \subseteq \text{supp } M$.

So $\dim R/\emptyset_0 < \dim M$ by hypothesis.

Hence $r \leq \dim M - 1 \implies (2)$.

To prove (3), note $xM = \text{Im } M/xM$. Form two exact sequences.
0 \to K \to M \to xM \to 0
0 \to xM \to M \to M/xM \to 0.

Thus \( d(K) \leq d(M) \), \( d(xM) \leq d(N) \).

Also
\[
P_2(K, n) + P_2(xM, n) - P_2(M, n)
\]
and
\[
P_2(xM, n) + P_2(M/xM, n) - P_2(M, n)
\]
are of degree \( \leq d - 1 \).

So their difference is too. Thus get (3).

Thm. (Dimension). \( R \) Noetherian, \( M \neq 0 \) a f.g. semi-local module. Then
\[
\dim M = d(M) = s(M) < \infty.
\]

Pf. Let's prove a cycle of inequalities.
First let us show \( \dim M \leq d(M) \). We proceed by induction in \( d(M) \). Suppose \( d(M) = 0 \).
Then \( \ell(M/M_{m+1}M) \) stabilizes. So \( m^n M = m^{n+1} M \)
for some \( n \). Hence \( m^n M = 0 \) by Nakayama lemma applied over the semi-local ring
\( R/\text{Ann}(M) \). So \( \ell(M) < \infty \). So \( \dim M = 0 \). (exercise above)

Suppose \( d(M) \geq 1 \). Then \( \dim R/M_{\phi_0} = \dim M \)
for some \( \phi_0 \in \text{supp} M \). Then \( \phi_0 \) is minimal.
So \( \phi_0 \in \text{Ass } M \). Hence \( M \) has a submodule
from \( R/\phi_0 \). Further, \( d(N) \leq d(M) \).
Take a chain of primes $p_0 \neq p_r \in \text{Supp} \ N$. If $r = 0$ then $r \leq d(M)$. Suppose $r = 1$. Then we have $x \notin \mathfrak{p}_1 \setminus p_0$. Also since $p_0$ is not maximal for each maximal ideal $n \in \text{Supp} \ N$, there exists $x_n \in n \setminus p_0$. Set $x = x_1 \Pi x_n$.

Then $x \in (p_1 \cap n) \setminus p_0$. Then $p_1 \neq p_0$ lies in $\text{Supp} \ N \setminus N(\langle x \rangle)$, which is $\text{Supp} (N/xN)$. So $r - 1 \leq \dim (N/xN)$.

However, $M$ is injective on $N$ as $N \cong R/p_0$ and $x \notin p_0$. So by previous lemma, we get $d(N/xN) \leq d(N) - 1$.

But $d(N) \leq d(M) = d(\mathfrak{p}_1)$ so $\dim (N/xN) \leq d(N/xN)$ by the induction hypothesis. Thus $r \leq d(M)$.

Thus $\dim M \leq d(M)$.

Second, let's prove $d(M) \leq s(M)$. Let $g$ be a prime ideal of $M$ with $s(M)$ generators. Then $d(M) = \deg P_g(M, n)$, but $\deg P_g(M, n) \leq s(M)$.

Finally, let's prove that $s(M) \leq \dim M$.

Let $r := \dim M$, which is finite since $r \leq d(M)$.

The proof is by induction on $r$.

If $r = 0$ then $M$ is of finite length, so $s(M) = 0$ and there is nothing to prove.
- 7 -

Suppose \( r \leq 1 \). Let \( P_1, \ldots, P_n \) be the primes of \( \text{Supp} M \) with \( \dim R/P_i = r \). No \( P_i \) is maximal as \( r \leq 1 \). So \( m \notin P_i \). Hence by prime avoidance \( \exists x \in m \) s.t. \( x \notin P_i \) for any \( i \).

So, the previous lemma yields \( s(M) \leq s(M/xM) + 1 \), and \( \dim (M/xM) + 1 \leq r \).

By id. hypoth, \( s(M/xM) \leq \dim M/xM \).

So \( s(M) \leq s(M/xM) + 1 \leq \dim M/xM + 1 \leq \dim M = r \), as desired. \( \Box \)

**Corollary.** \( R \) Noetherian, \( M \) a Noetherian semilocal module, \( x \in \text{rad} (M) \). Then \( \dim (M/xM) \geq \dim M - 1 \), with equality if \( x \notin \mathfrak{p} \) for \( \mathfrak{p} \in \text{Supp} M \) with \( \dim (R/\mathfrak{p}) = \dim M \).

Equality holds if \( x \notin 2 \text{div}(M) \).

**Pf.** By lemma above, we have \( s(M/xM) \geq s(M) - 1 \). So by this the asserted inequality holds. If the cond is satisfied, lemma above gives the opposite ineq, so equality. Finally, if \( x \notin 2 \text{div}(M) \), then \( x \notin \mathfrak{p} \) for \( \mathfrak{p} \in \text{Supp} M \) with \( \dim (R/\mathfrak{p}) = \dim M \) (as \( \mathfrak{p} \) is an assm. prime).
The height of $\mathfrak{p}$, $\text{ht}(\mathfrak{p})$, is defined by the formula:

$$\text{ht}(\mathfrak{p}) = \sup \{|\mathfrak{p}| : \exists \text{ a chain of primes } \mathfrak{p}_0 \supsetneq \cdots \supsetneq \mathfrak{p}_r = \mathfrak{p}\}.$$  

Thus $$\text{ht}(\mathfrak{p}) = \dim(R_{\mathfrak{p}}).$$  

(by bij. correspondence of primes under localization).

If $\text{ht}(\mathfrak{p}) = h$, call $\mathfrak{p}$ a height $h$ prime.

Cor. If $\mathfrak{p}$ is a height $h$ prime.

Then $\text{ht}(\mathfrak{p}) \leq r \iff \mathfrak{p}$ is a minimal (assoc.) prime of some ideal generated by $r$ elements.

(Insight: suppose we have a subvariety $X$ in $\mathbb{C}^n$ of codimension $r$. Then we may not necessarily describe $X$ by $r$ equations, but we can always describe it as an irreducible component of the set of solutions of $r$ equations.)

If $\mathfrak{p}$ is a minimal cont. ideal of height $r$, then $\mathfrak{p}$ is of the form $\mathfrak{q}R_{\mathfrak{p}}$, where $\mathfrak{q}$ is prime in $R$. 

with or \( \emptyset \neq \mathfrak{p} \neq \mathfrak{p} \). So \( q = \mathfrak{p} \) as \( \mathfrak{p} \) is min.

Hence \( \sqrt{\mathfrak{p}} = \sqrt{\mathfrak{p}} \mathfrak{p} \). By Steinwellstellem
sat. So \( r \geq s(\mathfrak{p}) \) by above (or \( \mathfrak{p} \) is a parameter ideal). But \( s(\mathfrak{p}) = \dim \mathfrak{p} \mathfrak{p} \) and \( \dim \mathfrak{p} \mathfrak{p} = h(\mathfrak{p}) \). So \( h(\mathfrak{p}) \leq r \).

Conversely, assume \( h(\mathfrak{p}) \leq r \). Then \( \mathfrak{p} \) has a parameter ideal \( \mathfrak{p} \mathfrak{p} \) generated by \( r \) elements. Say, \( y_1, \ldots, y_r \), as \( h(\mathfrak{p}) = \dim \mathfrak{p} \mathfrak{p} = s(\mathfrak{p}) \). Say
\[
y_i = \frac{x_i}{s_i}, \quad s_i \notin \mathfrak{p}. \quad \text{set } \alpha = (x_1, \ldots, x_r).
\]
Supp. there is a prime \( q \) with or \( q \mathfrak{p} \).

Then \( b = \alpha \mathfrak{p} \subset q \mathfrak{p} \mathfrak{p} \subset \mathfrak{p} \mathfrak{p} \) and \( q \mathfrak{p} \mathfrak{p} \) is prime by (11.20) (2). But \( \sqrt{b} = \mathfrak{p} \mathfrak{p} \).

So \( q = \mathfrak{p} \) by (11.20) (2). Then \( \mathfrak{p} \) is min.

cont. \( \alpha \), which is gen. by \( r \) elements. \( \square \).

Thm. (Krull Principal ideal thm).

R Noetherian, \( x \in R \), \( \mathfrak{p} \) a minimal prime of \( \langle x \rangle \).
If \( x \notin 2 \text{div}(R) \) then \( h(\mathfrak{p}) = 1 \).
(All components of \( R \) hypersurface have codim 1).

Pf. By Cor above, \( h(\mathfrak{p}) \leq 1 \). But if \( h(\mathfrak{p}) = 0 \),
then \( \mathfrak{p} \) is min. since \( \mathfrak{p} \) is

cont. \( x \), which is gen. by \( 1 \) elements. \( \square \).
Cor. A, B Noetherian, m, n their max. ideals.
\[ \psi : A \to B \] local homom. Then
\[ \dim B \leq \dim A + \dim B/mB \]
with equality if \( B \) is flat over \( A \). \( \psi^{-1}(n) = m \).
(This can be interpreted as thm on dim of fibers.)

Pf. Let \( s = \dim A \). \( \exists \) a param. ideal of \( A \) generated by \( s \) elements. So \( m/2 \) is nilpotent in \( R/q \). Hence \( mB/qB \) is nilpotent in \( B/qB \Rightarrow \dim B/mB = \dim B/qB \). But \( \dim B/qB \geq \dim B - s \) by cor. above. Thus the inequality holds.

Assume \( B \) is flat over \( A \).
Let \( p \supseteq mB \) be a prime with \( \dim(B/p) = \dim(B/mB) \). Then \( \dim B \geq \dim B/p + \text{ht}(p) \) by containment of chains of primes. Thus it suffices to show that \( \text{ht}(p) \geq \dim A \).
As \( n \supseteq p \supseteq mB \) (as \( \psi \) is local), we have \( \psi^{-1}(p) = m \). Since \( B \) is flat over \( A \), going down for flat algebras gives a chain of primes of \( B \) lying over any given chain in \( A \). So \( \text{ht}(p) \geq \dim A \).
Regular local rings.

A Noetherian local ring $\dim = r$. Say $A$ is regular if its maximal ideal is generated by $r$ elements. Then any $r$ generators are said to form a regular system of parameters. So we have $R$ regular $\iff r = \dim \left( \frac{R}{m} \right)$ (in general, $r \leq \dim \left( \frac{R}{m^2} \right)$).

Ex. A field is a regular local ring if $\dim = 0$, and vice versa.

An ex. of a regular local ring of a given dimension $n$ is $k[x_1, \ldots, x_n]_m$ or $k[[x_1, \ldots, x_n]]$.

Lemma. A a Noetherian semilocal of dimension $d$, $m$ its parameter ideal. Then
\begin{equation}
\deg h(G^A, n) = r - 1.
\end{equation}

Proof. $\deg h(G^A, n)$ is 1 less than the order of the pole of $H(G^A, t)$. But this order equals $d(A)$. Also $d(A) = r$ by dim them. B

Prop. A Noetherian local of $\dim = r$, $m$ maximal ideal. Then $A$ is regular if and only if $G^A$ is a polynomial ring. If so then the number of variables is $r$. 
Pf. Say $GA$ is a polyh. ring in $s$ variables.

By the above, $\deg h(GA, n) = s - 1$. So $s = r$ by the above lemma. So $A$ is regular by definition ($r = \dim (\mathbb{M}/m^2)$).

Conversely, assume $A$ regular. Let $x_1, \ldots, x_r$ be a regular sop, and $x_i \in \mathbb{M}/m^2$ the residue of $x_i$. Let $h = \mathbb{M}/m$, $P = k[x_1, \ldots, x_r]$ be the polyh. ring. We have a homomorphism $\varphi: P \to GA$ s.t. $\varphi(x_i) = x_i'$. Then $\varphi$ is surjective as $x_i'$ generate $GA$. Let $\sigma = \ker \varphi$.

Let $P = \oplus \sigma_n$ be the grading by total degree. Then $\varphi$ preserves the grading of $P$ and $GA$. So $\sigma$ inherits a grading $\sigma = \oplus \sigma_n$. So $\forall n \geq 0$ have a canonical exact sequence

$$0 \to \sigma_n \to P_n \to \mathbb{M}/m^{n+1} \to 0$$

Suppose $\sigma_0 \neq 0$. Then $\exists$ a nonzero $f \in \sigma_m$ for some $m$. Take $n \geq m$. Then $P_n - m \sigma_n$.

Since $P$ is a domain, $P_n - m \leq P_n - m F$.

So we get

$$\dim (\mathbb{M}/m^{n+1}) = \dim (P_n) - \dim (\sigma_n)$$

$$\leq \dim P_n - \dim P_{n-m} = \left( \frac{r-1+n}{r-1} \right) - \left( \frac{r-1+n-m}{r-1} \right)$$
The expr. on the right is a poly of degree r-2.

On the other hand, \( \dim \left( \frac{m^n}{m^{n+1}} \right) = \delta(A, \mathfrak{m}) \)
for \( n \gg 0 \), and \( \deg h(\mathfrak{m}A, r) = r-1 \) by above. That so we have a contrad. So \( \mathfrak{m} \mathfrak{z} = 0 \), and \( \mathfrak{p} \) is an isom.

Thm. A regular local ring is a domain.

Pf. Use ind. in \( r = \dim A \). If \( r = 0 \), \( A \) is a field, so a domain. Assume \( r \geq 1 \).

Let \( \mathfrak{x} \) be a member of a local regular sop. Then \( A/\mathfrak{x}A \) is regular of dim \( r-1 \) (exercise). By ind, \( A/\mathfrak{x}A \) is a domain.

So \( \langle \mathfrak{x} \rangle \) is prime. Hence \( A \) is a domain.

Alternatively: let \( f, g = 0 \). Take supp \( f \in \mathfrak{m}^k \) but not in \( \mathfrak{x} \mathfrak{k} \), and \( g \in \mathfrak{m}^s \) but not \( \mathfrak{m}^{s-1} \). Let \( \overline{f}, \overline{g} \) images in \( \mathfrak{m}^k A, \mathfrak{m}^s A \). Then \( \overline{f} \overline{g} \neq 0 \), \( \overline{f}, \overline{g} \neq 0 \) \( \Rightarrow \) (as \( A/\mathfrak{x} \) is a poly, ring).

Lemma A local, \( \mathfrak{m} \) max, \( \mathfrak{p} \) proper ideal.

let \( n = \mathfrak{m}^a \), \( k = A/\mathfrak{m} \). Then we have an exact seq:

\[ 0 \rightarrow \mathfrak{m}^2 + \mathfrak{p}/\mathfrak{m}^2 \rightarrow \mathfrak{m}/\mathfrak{m}^2 \rightarrow \mathfrak{m}/\mathfrak{m}^2 \rightarrow 0. \]
Prop. Let $A$ be a regular local ring of dim = $r$, or an ideal. Set $B = A/\alpha$, and assume $B$ regular of dim $r-s$. Then $B$ is generated by $s$ elements, and such $s$ elements form a part of a regular system of parameters.

Rem. This is a version of implicit function theorem in multivariable calc.

Pt. By lemma above, $\dim (\mathfrak{m}^2 + \alpha/\mathfrak{m}^2) = s$. Hence any set of generators of $\mathfrak{m}$ includes a member of a reg. sys. of $\mathfrak{m}$. Let $\alpha$ be the ideal in $A$, then generate.

Then $A/\alpha$ is reg of dim $r-s$ by exer. above. By this above, both $A/\alpha$ and $B$ are domains of dim $r-s$. But if we impose an eqn. in a domain, dim drops.