Lecture 21: Dimension.

The dimension of M

$$\dim M = \sup\{r \mid \exists \text{ a chain of primes } \mathfrak{p}_0 \subsetneqq \mathfrak{p}_1 \subsetneqq \cdots \subsetneqq \mathfrak{p}_r \text{ in } \operatorname{Supp} M\}.$$

Assume R Noetherian, M f. gen. Then M has fin. many min assoc. primes. They are also min primes in $\operatorname{Supp} M$. Thus

$$\dim M = \max\{\dim R/\mathfrak{p}_0 \mid \mathfrak{p}_0 \in \operatorname{Supp} M \text{ minimal}\}.$$

Parameters. R a ring, $M \neq 0$ R-module. Denote $\bigcap$ of max ideals in $\operatorname{Supp} M$ as $\operatorname{rad} M$, and call it the radical of M. If there are fin. many such max. ideals, call M semilocal. Call $\mathfrak{q} \subset \operatorname{rad} M$ a <u>parameter ideal</u> for M if $M/\mathfrak{q}M$ is Artinian.

$\boxed{\text{Ex: } M = R = (\mathbb{C}[x_1, .., x_e]/I)_{\mathfrak{m}}. \text{ Then } \operatorname{rad} M = \mathfrak{m}. \; \mathfrak{q} \subset \mathfrak{m} \text{ param if } R/\mathfrak{q}R \text{ Artinian (f. dim}/\mathbb{C}\text{), e.g. local coord.}}$

Assume M is f. gen. Then $\operatorname{Supp} M = V(\operatorname{Ann} M)$. So M is semilocal $\iff$ $R/\operatorname{Ann} M$ is a semilocal ring.

Assume R Noetherian. So M Noetherian. Fix $\mathfrak{q} \subset R$. Then $M/\mathfrak{q}M$ is Artinian $\iff$ $\ell(M/\mathfrak{q}M) < \infty$. But $\ell(M/\mathfrak{q}M) < \infty \iff \operatorname{supp}(M/\mathfrak{q}M)$ consists of fin. many max. ideals, ~~i.e. iff~~

Also

$\operatorname{supp}(M/qM) = \operatorname{Supp} M \cap V(q) = V(\operatorname{Ann} M) \cap V(q)$

$= V(\operatorname{Ann} M + q)$.

Set $q' = \operatorname{Ann} M + q$.

Then M/qM is Artinian $\Leftrightarrow$ $V(q')$ consists of f.m. max ideals

$\Longleftrightarrow$ (Akizuki-Hopkins) R/q' Artinian.

So by exer 19.18 R/q' Artinian

$\Longleftrightarrow$ q' cont a product of max ideals each containing q'.

So each lies in $\operatorname{Supp} M$, hence contains $\operatorname{rad}(M)$. Thus R/q' Artinian $\Longrightarrow$ $q' \supset m^n$ for some n.

Set $m = \operatorname{rad} M$. Assume M semilocal, so $\operatorname{Supp} M$ has f. many max ideals. Then their product is cont in m. Thus, conversely, if $q' \supset m^n$ for some $n > 0$, then R/q' Artinian. Thus:

<u>Prop:</u> q is a param ideal $\Longleftrightarrow$

$m \supset q' \supset m^n$ for some n,

or ~~by~~ iff $m = \sqrt{q'} \Leftrightarrow V(m) = V(q')$.

In partic, m^n is a param ideal $\forall n$.

Assume q is a parameter ideal. (m-radical)
Then the Hilbert-Samuel polyn. $P_q(M,n)$ exists. Also $P_m(M,n)$ exists, and the two polyn. have the same degree. (by exercise above), since $m=\sqrt{q'}$ and $P_{q'}=P_q$. Thus the degree is the same for any parameter ideal. Denote this common degree by $d(M)$.

(Basic ex: R local ring of a variety, M f.g. R-module).

Alternatively, $d(M)$ = order of pole at 1 of $H(G^0M,t)$ (it is 1 less than that of $P_q(M,t)$)

Let $s(M)$ be the smallest s such that $\exists\, x_1,\dots,x_s \in M$ with $\ell(M/\langle x_1,\dots,x_s\rangle M)<\infty$. (if $\ell(M)<\infty$ then $s(M)=0$). We say that $x_1,\dots,x_s \in m$ form a system of parameters for M if $s=s(M)$ and $\ell(M/\langle x_1,\dots,x_s\rangle M)$ holds. Such a sop generates a parameter ideal.

<u>Lemma</u>. R Noeth, $M \neq 0$ a Noeth $(= \text{f.g.})$ semilocal module, q param. ideal of M, $x \in \operatorname{rad} M$. Let $K = \operatorname{Ker}(M \xrightarrow{\mu_x} M)$.

(1) $s(M) \leq s(M/xM) + 1$.

(2) $\dim(M/xM) \leq \dim M - 1$ if $x \notin \mathfrak{p}$ for any $\mathfrak{p} \in \operatorname{Supp} M$ with $\dim R/\mathfrak{p} = \dim M$.

(3) $\deg\left(P_q(K, n) - P_q(M/xM, n)\right) \leq d(M) - 1$.

<u>Pf.</u> For (1), set $s = s(M/xM)$. $\exists x_1, \dots, x_s \in \operatorname{rad}(M/xM)$ with $\ell(M/\langle x, x_1, \dots, x_s\rangle M) < \infty$.

Also $\operatorname{supp} M/xM = \operatorname{supp} M \cap V(\langle x\rangle)$. But $x \in \operatorname{rad} M$, so $\operatorname{supp} M/xM$ and $\operatorname{supp} M$ have the same max. ideals. So $\operatorname{rad}(M/xM) = \operatorname{rad} M$.

Thus $s(M) \leq s+1 \Rightarrow (1)$

For (2) Take chain $\mathfrak{p}_0 \subsetneq \cdots \subsetneq \mathfrak{p}_r$ in $\operatorname{Supp} M/xM$.

Again $\operatorname{Supp} M/xM = \operatorname{Supp} M \cap V((x))$. So $x \in \mathfrak{p}_0 \in \operatorname{Supp} M$.

So $\dim R/\mathfrak{p}_0 < \dim M$ by hypothesis.

Hence $r \leq \dim M - 1 \Rightarrow (2)$.

To prove (3), note $xM = \operatorname{Im} \mu_x$. Form two exact sequences:

$$0 \to K \to M \to xM \to 0$$

$$0 \to xM \to M \to M/xM \to 0.$$

Thus $d(K) \le d(M)$, $d(xM) \le d(M)$.

Also $P_q(K,n) + P_q(xM,n) - P_q(M,n)$

and

$P_q(xM,n) + P_q(M/xM, n) - P_q(M,n)$

are of degree $\le d(M)-1$.

So their difference is too. Thus get (3).

<u>Thm.</u> (Dimension). R Noetherian, $M \ne 0$ a f.g. semilocal module. Then

$$\dim M = d(M) = s(M) < \infty.$$

<u>Pf.</u> Let's prove a cycle of inequalities. First let us show $\dim M \le d(M)$. We proceed by induction in $d(M)$. Suppose $d(M) = 0$. Then $\ell(M/m^n M)$ stabilizes. So $m^n M = m^{n+1} M$ for some n. Hence $m^n M = 0$ by Nakayama lemma applied over the semilocal ring $R/\mathrm{Ann}(M)$. So $\ell(M) < \infty$. So $\dim M = 0$. (exercise above).

Suppose $d(M) \ge 1$. Then $\dim R/\mathfrak{p}_0 = \dim M$ for some $\mathfrak{p}_0 \in \mathrm{supp}\, M$. Then $\mathfrak{p}_0$ is minimal. So $\mathfrak{p}_0 \in \mathrm{Ass}\, M$. Hence M has a submod isom to $R/\mathfrak{p}_0$. Further, $d(N) \le d(M)$.

Take a chain of primes $\mathfrak{p}_0 \subsetneq \cdots \subsetneq \mathfrak{p}_r$ in Supp N.
If $r=0$ then $r \leq d(M)$. Suppose $r \geq 1$. Then have $x \notin \mathfrak{p}_1 - \mathfrak{p}_0$. Also since $\mathfrak{p}_0$ is not maximal, for each maximal ideal $n \in \operatorname{Supp} M$, $\exists\, x_n \in n - \mathfrak{p}_0$. Set $x = x_1 \prod x_n$.
Then $x \in (\mathfrak{p}_1 \cap m) \setminus \mathfrak{p}_0$. Then $\mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_r$ lies in $\operatorname{Supp} N \cap V(\langle x \rangle)$, which is $\operatorname{Supp}(N/xN)$. So $r-1 \leq \dim(N/xN)$.
However, μ_x is injective on N as $N \cong R/\mathfrak{p}_0$ and $x \notin \mathfrak{p}_0$. So by previous lemma, we get $d(N/xN) \leq d(N) - 1$.
But $d(N) \leq d(M)$ ~~⇒ d~~ so $\dim(N/xN) \leq d(N/xN)$ by the ind. hypothesis. Thus $r \leq d(M)$.
Thus $\dim M \leq d(M)$.

Second, let's prove $d(M) \leq s(M)$. Let q be a param ideal of M with $s(M)$ generators. Then $d(M) = \deg P_q(M,n)$, but $\deg P_q(M,n) \leq s(M)$
$\Rightarrow d(M) \leq s(M)$.

Finally, let's prove that $s(M) \leq \dim M$.
Let $r := \dim M$, which is finite since $r \leq d(M)$.
The proof is by induction on r.
If $r=0$ then M is of finite length so $s(M)=0$ and there is nothing to prove.

Suppose $r \geq 1$. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_k$ be the primes of $\operatorname{Supp} M$ with $\dim R/\mathfrak{p}_i = r$. No $\mathfrak{p}_i$ is maximal as $r \geq 1$. So $\mathfrak{m} \not\subset \mathfrak{p}_i \ \forall i$. Hence by prime avoidance $\exists x \in \mathfrak{m}$ s.t. $x \notin \mathfrak{p}_i$ for any i.

So the previous lemma yields

$s(M) \leq s(M/xM) + 1$, and $\dim (M/xM) + 1 \leq r$.

By ind. hypoth, $s(M/xM) \leq \dim M/xM$.

So $s(M) \leq s(M/xM) + 1 \leq \dim M/xM + 1 \leq \dim M = r$, as desired. $\square$

<u>Corollary</u>. R Noetherian, $M \neq 0$ a Noetherian semilocal module, $x \in \operatorname{rad}(M)$. Then $\dim (M/xM) \geq \dim M - 1$, with equality if $x \notin \mathfrak{p}$ for $\mathfrak{p} \in \operatorname{Supp} M$ with $\dim (R/\mathfrak{p}) = \dim M$. Equality holds if $x \notin \operatorname{zdiv}(M)$

<u>Pf.</u> By lemma above, we have $s(M/xM) \geq s(M) - 1$. So by thm the asserted inequality holds. If the cond is satisfied, lemma above gives the opposite ineq, so equality. Finally, if $x \notin \operatorname{z.div} M$, then $x \notin \mathfrak{p}$ for $\mathfrak{p} \in \operatorname{Supp} M$ with $\dim (R/\mathfrak{p}) = \dim M$ (as $\mathfrak{p}$ is an assoc prime). $\square$

__Height__. $R \supset \mathfrak{p}$ prime. The __height__ of $\mathfrak{p}$, $ht(\mathfrak{p})$, is defined by the formula:

$$ht(\mathfrak{p}) = \sup\{r \mid \exists \text{ a chain of primes}$$

$$\mathfrak{p}_0 \subsetneq \cdots \subsetneq \mathfrak{p}_r = \mathfrak{p}.$$

Thus

$$ht(\mathfrak{p}) = \dim(R_{\mathfrak{p}}).$$

(by bij. correspondence of primes under localization).

If $ht(\mathfrak{p}) = h$, call $\mathfrak{p}$ a height h prime.

__Cor.__ R Noeth, $\mathfrak{p} \subset R$ prime. Then $ht(\mathfrak{p}) \leq r \iff \mathfrak{p}$ is a minimal (assoc.) prime of some ideal generated by r elements.

(Intuition: suppose we have a subvariety X in $\mathbb{C}^n$ of codimension r. Then we may not necessarily describe X by r equations but can always describe it as an irreducible component of the set of solutions of r equations.)

__Pf.__ Ass. $\mathfrak{p}$ minimal cont ideal $\mathfrak{a}$ gen. by r elements. Now, any prime in $R_{\mathfrak{p}}$ cont. $\mathfrak{a}R_{\mathfrak{p}}$ is of the form $\mathfrak{q}R_{\mathfrak{p}}$ where $\mathfrak{q}$ is prime in R

with $\mathfrak{a} \subset \mathfrak{q} \subset \mathfrak{p}$. so $\mathfrak{q} = \mathfrak{p}$ as $\mathfrak{p}$ is min.
Hence $\mathfrak{p}R_\mathfrak{p} = \sqrt{\mathfrak{a}R_\mathfrak{p}}$ by Scheinnullstellensatz. So $r \geq s(R_\mathfrak{p})$ by above ($\mathfrak{a}R_\mathfrak{p}$ is a parameter ideal). But $s(R_\mathfrak{p}) = \dim R_\mathfrak{p}$ and $\dim R_\mathfrak{p} = ht(\mathfrak{p})$. So $ht(\mathfrak{p}) \leq r$.

Conversely, assume $ht(\mathfrak{p}) \leq r$. Then $R_\mathfrak{p}$ has a parameter ideal $\mathfrak{b}$ generated by r elements, say, $y_1, \dots, y_r$, as $ht(\mathfrak{p}) = \dim R_\mathfrak{p} = s(R_\mathfrak{p})$. Say $y_i = \frac{x_i}{s_i}$, $s_i \notin \mathfrak{p}$. Set $\mathfrak{a} = (x_1, \dots, x_r)$.
Supp. there is a prime $\mathfrak{q}$ with $\mathfrak{a} \subset \mathfrak{q} \subset \mathfrak{p}$.
Then $\mathfrak{b} = \mathfrak{a}R_\mathfrak{p} \subset \mathfrak{q}R_\mathfrak{p} \subset \mathfrak{p}R_\mathfrak{p}$ and $\mathfrak{q}R_\mathfrak{p}$ is prime by (11.20)(2). But $\sqrt{\mathfrak{b}} = \mathfrak{p}R_\mathfrak{p}$
So $\mathfrak{q} = \mathfrak{p}$ by (11.20)(2). Thus $\mathfrak{p}$ is min. cont $\mathfrak{a}$, which is gen. by r elements. □.

<u>Thm</u>. (Krull Principal ideal thm).
R Noetherian, $x \in R$, $\mathfrak{p}$ a minimal prime of $\langle x \rangle$
If $x \notin \text{zdiv}(R)$ then $ht(\mathfrak{p}) = 1$.

(All components of a hypersurface have codim 1).

<u>Pf</u>. By Cor above, $ht(\mathfrak{p}) \leq 1$. But if $ht(\mathfrak{p}) = 0$, then $\mathfrak{p}$ is min. ~~and max~~, so consists of zero div, as proved

Cor. A, B Noetherian, local $\mathfrak{m}, \mathfrak{n}$ their max. ideals. $\varphi : A \to B$ local homom. Then

$$\dim B \leq \dim A + \dim(B/\mathfrak{m}B) \quad *$$

with equality if B is flat over A. (recall a local homom is $\varphi^{-1}(\mathfrak{n}) = \mathfrak{m}$).

(this can be interpreted as thm on dim of fibers).

Pf. Set $s = \dim A$. $\exists$ a param. ideal q of A generated by s elements. So $\mathfrak{m}/q$ is nilpotent in R/q. Hence $\mathfrak{m}B/qB$ is nilpotent in $B/qB \Rightarrow \dim B/\mathfrak{m}B = \dim B/qB$. But $\dim B/qB \geq \dim B - s$ by cor. above. Thus the inequality holds.

Assume B is flat over A. Let $\mathfrak{p} \supset \mathfrak{m}B$ be a prime with ~~dim~~ $\dim(B/\mathfrak{p}) = \dim(B/\mathfrak{m}B)$. Then $\dim B \geq \dim B/\mathfrak{p} + ht(\mathfrak{p})$ by concatenation of chains of primes. Thus it suffices to show that $ht(\mathfrak{p}) \geq \dim A$. As $\mathfrak{n} \supset \mathfrak{p} \supset \mathfrak{m}B$ (as φ is local), we have $\underset{\mathfrak{m}}{\varphi^{-1}(\mathfrak{n})} \supset \varphi^{-1}(\mathfrak{p}) \supset \underset{\mathfrak{m}}{\varphi^{-1}(\mathfrak{m}B)}$ $\varphi^{-1}(\mathfrak{p}) = \mathfrak{m}$. Since B is flat over A, going down for flat algebras gives a chain of primes of B lying over any given chain in ~~A~~ A descending from $\mathfrak{p}$. So $ht\,\mathfrak{p} \geq \dim A$, as claim

Regular local rings.

A Noetherian local $\dim = r$. Say A regular if its max ideal is generated by r elem. Then any r generators are said to form a regular system of parameters.

So we have R regular $\Leftrightarrow r = \dim(m/m^2)$ (in general $r \leq \dim m/m^2$).

Ex. A field is a regular local ring of $\dim = 0$, and vice versa.

An ex. of a regular local ring of a given dim n is $k[x_1, \ldots, x_n]_m$ or $k[[x_1, \ldots, x_n]]$.

Lemma A a Noetherian semilocal of dim r, q its parameter ideal. Then $\deg h(G^{\cdot}A, n) = r - 1$.

Pf. $\deg h(G^{\cdot}A, n)$ is 1 less than the order of the pole of $H(G^{\cdot}A, t)$. But this order equals $d(A)$. Also $d(A) = r$ by dim thm. $\square$

Prop. A Noetherian local of $\dim = r$, m max ideal. Then A is regular iff $G^{\cdot}A$ is a poly n. ring. If so then the number of variables is r.

<u>Pf.</u> Say $G^{\cdot}A$ is a polyn. ring in s variables (prev. lecture)
By the above, $\deg h(G^{\cdot}A, n) = s-1$. So $s = r$
by the above lemma. So A is regular
by definition ($r = \dim(m/m^2)$)

Conversely, assume A regular. Let $x_1, .., x_r$
be a regular sop, And $x_i' \in m/m^2$ the residue
of x_i. Let $k = A/m$, $P = k[X_1, .., X_r]$ be
the polyn. ring. We have a homom.
$\varphi : P \to G^{\cdot}A$ s.t. $\varphi(X_i) = x_i'$. Then φ is
surjective as x_i' generate $G^{\cdot}A$. Let $\mathfrak{a} = \ker\varphi$.
Let $P = \oplus P_n$ be the grading by total
degree. Then φ preserves the grading of
P and $G^{\cdot}A$. So $\mathfrak{a}$ inherits a grading:
$\mathfrak{a} = \oplus \mathfrak{a}_n$. So $\forall n \geq 0$ have a canonical exact
sequence

$$0 \to \mathfrak{a}_n \to P_n \to m^n/m^{n+1} \to 0.$$

Suppose $\mathfrak{a} \neq 0$. Then $\exists$ a nonzero $f \in \mathfrak{a}_m$
for some m. Take $n \geq m$. Then $P_{n-m} f \subset \mathfrak{a}_n$.
Since P is a domain, $P_{n-m} \cong P_{n-m} f$.
So we get

$$\dim(m^n/m^{n+1}) = \dim(P_n) - \dim(\mathfrak{a}_n)$$

$$\leq \dim P_n - \dim P_{n-m} = \binom{r-1+n}{r-1} - \binom{r-1+n-m}{r-1}.$$

The expr. on the right is a polyn of degree $r-2$.

On the other hand, $\dim(m^n/m^{n+1}) = h(G^{\cdot}A, n)$ for $n \gg 0$, and $\deg h(G^{\cdot}A, r) = r-1$ by above. ~~But~~ So we have a contrad. So $\mathfrak{a} = 0$, and φ is an isom.

<u>Thm.</u> A regular local ring is a domain.

<u>Pf.</u> Use ind. in $r = \dim A$. If $r=0$, A is a field, so a domain. Assume $r \geq 1$. Let x be a member of a ~~local~~ regular sop. Then A/xA is regular of dim $r-1$ (exercise). By ind, $A/\langle x \rangle$ is a domain. So $\langle x \rangle$ is prime. Hence A is a domain.

<u>Alternatively</u>: Let $fg = 0$. $f, g \neq 0$. ~~Take~~ Supp $f \in m^k$ but not in f^{k-1}, and $g \in m^s$ but not m^{s-1}. Let $\bar{f}, \bar{g}$ images in G^kA, G^sA. Then $\bar{f}\bar{g} \neq 0$, $\bar{f}, \bar{g} \neq 0 \Rightarrow \Leftarrow$ (as $G^{\cdot}A$ is a poly. ring). $\square$

<u>Lemma</u> A local, m max, $\mathfrak{a}$ proper ideal. Set $n = m/\mathfrak{a}$, $k = A/m$. Then we have an exact seq.

$$0 \to m^2 + \mathfrak{a}/m^2 \to m/m^2 \to n/n^2 \to 0.$$

Pf. easy

Prop. Let A be a regular local ring of $\dim = r$, $\mathfrak{a}$ an ideal. Set $B = A/\mathfrak{a}$, and assume B regular of dim $r-s$. Then $\mathfrak{a}$ is generated by s elements, and such s elements form a part of a regular system of parameters.

Rem. This is a version of implicit function thm in multivariable calc.

Pf. By lemma above, $\dim\left((m^2+\mathfrak{a})/m^2\right) = s$. Hence any set of generators of $\mathfrak{a}$ includes s member of a reg. sys of par. of A. Let $\mathfrak{b}$ be the ideal they generate. Then $A/\mathfrak{b}$ is reg of dim $r-s$ by exer. above. By thm above, both $A/\mathfrak{b}$ and B are domains of dim $r-s$. But if we impose an eqn. in a domain, dim drops.