Lecture 19. Length

R a ring, M an R-module.

Def. M *simple* if $M \neq 0$ and its only proper submodule is 0.

A chain $M = M_0 \supset M_1 \supset \cdots \supset M_n = 0$ is a *comp. series* of length ~~n~~ m if M_{i-1}/M_i are simple.

Length of M $\ell(M)$ is $\inf\{m \mid M$ has a comp. series of length $m\}$.

If M has no comp. series, then $\ell(M) = \infty$. Also $\ell(M) = 0 \Leftrightarrow M = 0$.

Ex. If R is a field, then $\ell(M) = \dim M$.

Thm. (Jordan – Holder). Suppose M has a comp. series $M = M_0 \supset M_1 \supset \cdots \supset M_m = 0$. Then any chain of submodules can be refined to a comp. series, and every composition series has the same length $\ell(M)$.

Also $\operatorname{Supp} M = \{ m \in \operatorname{Spec} R \mid m = \operatorname{Ann}(M_{i-1}/M_i)$ for some $i\}$. the $m \in \operatorname{Supp} M$ are maximal; there is a canonical isom.

$$M \cong \prod_{m \in \operatorname{Supp} M} M_m$$

and $\ell(M_m)$ is the number of i with $m = \operatorname{Ann}(M_{i-1}/M_i)$

Pf. Let M' be a proper submodule
Let us show $\ell(M') < \ell(M)$.
To do so, let $M_i' = M_i \cap M'$.
Then $M_{i-1}' \cap M_i = M_i'$. So $M_{i-1}'/M_i' \subset M_{i-1}/M_i$.
Since M_{i-1}/M_i simple, either $M_{i-1}'/M_i' = 0$ or $M_{i-1}'/M_i' = M_{i-1}/M_i$, so $M_{i-1}' + M_i = M_{i-1}$. If this holds and $M_i \subset M'$ then $M_{i-1} \subset M'$. So this can't hold for all i, otherwise $M = M'$.
So for some i, $M_{i-1}' = M_i'$, and we can omit one term. So get a ~~chain for~~ comp series for M' shoter thatn that for M. Can choose shortest for M, so get $\ell(M') < \ell(M)$.

Now for any chain $N_0 \supsetneq N_1 \supsetneq \cdots \supsetneq N_n = 0$ let us show that $n \le \ell(M)$, by induction in $\ell(M)$. If $\ell(M) = 0$, it's clear. Ass. $\ell(M) \ge 1$. If $n = 0$, clear. If $n \ge 1$ then $\ell(N_1) < \ell(M)$ so $n - 1 \le \ell(N_1)$ by ind. ass $\Rightarrow n \le \ell(M)$.
If N_{i-1}/N_i is not simple, then $\exists N'$ $N_{i-1} \supsetneq N' \supsetneq N_i$, and we can make the

chain longer. Repeating, we can refine the chain to a comp. series in at most $\ell(M)-n$ steps.

Supp. a given chain is a comp. series. then $\ell(M) \le n$. But also $\ell(M) \ge n$, so $\ell(M) = n$. The first assertion is proved.

To proceed, fix a prime $\mathfrak{p}$. Exactness of localization yields a chain

$$M_{\mathfrak{p}} = (M_0)_{\mathfrak{p}} \supset (M_1)_{\mathfrak{p}} \supset \cdots \supset (M_m)_{\mathfrak{p}} = 0.$$

Now consider a max ideal $\mathfrak{m}$. If $\mathfrak{p} = \mathfrak{m}$ then $(R/\mathfrak{m})_{\mathfrak{p}} = R/\mathfrak{m}$. If $\mathfrak{p} \ne \mathfrak{m}$, then $\exists s \in \mathfrak{m} \setminus \mathfrak{p}$, so $R/\mathfrak{m})_{\mathfrak{p}} = 0$. set $\mathfrak{m}_i = \mathrm{Ann}(M_{i-1}/M_i)$

So $M_{i-1}/M_i = R/\mathfrak{m}_i$, and $\mathfrak{m}_i$ is max. as M_{i-1}/M_i is simple.

Thus $\mathrm{Supp}\, M = \{\mathfrak{m}_1, \ldots, \mathfrak{m}_n\}$

as $(M_{i-1})_{\mathfrak{p}}/(M_i)_{\mathfrak{p}} = \begin{cases} 0, & \mathfrak{p} \ne \mathfrak{m}_i \\ M_{i-1}/M_i \simeq R/\mathfrak{m}_i, & \mathfrak{p} = \mathfrak{m}_i. \end{cases}$

If we omit duplicates we get comp. series ~~from~~ the $(M_i)_{\mathfrak{p}}$ with $M_{i-1}/M_i = R/\mathfrak{p}$. ($\mathfrak{p} = \mathfrak{m}_j$ for some j).

Thus the number of such i is $\ell(M_{\mathfrak{p}})$.

Finally, consider the canonical map $\varphi : M \to \prod_{m \in Supp(M)} M_m$. To prove that

φ is an isom, it suffices to prove that φ_p is for each p. (can check exactness at each max ideal).

Now, localiz commutes with finite products. So

$$\varphi_p : M_p \to \left(\prod_m M_m\right)_p = \prod_m (M_m)_p = M_p$$

Thus $\varphi_p = 1$. $\Box$.

<u>Cor.</u> M both artinian and Noetherian $\Leftrightarrow$ M has finite length.

<u>Pf.</u> Any chain $M \supset N_0 \supsetneq \cdots \supsetneq N_n = 0$ has length $n < \ell(M)$ by JH thm. So if $\ell(M) < \infty$ then M sat both acc and dcc.

Conversely, assume M is both Noetherian and Artinian. Form a chain as foll. Set $M_0 = M$. For $i \geq 1$, if $M_{i-1} \neq 0$, take max $M_i \subsetneq M_{i-1}$. (exists by maxc). By the dcc, the chain terminates. Then the chain is a comp. series.

<u>Thm.</u> (Additivity of length). $M \supset M'$. Then $\ell(M) = \ell(M') + \ell(M/M')$.

<u>Pf.</u> ~~Suppose~~ Any comp. ser. for M' and M/M' gives rise to a comp. series of M through M'. So $\ell(M') < \infty$, $\ell(M/M') < \infty$ $\Rightarrow \ell(M) = \ell(M') + \ell(M/M')$. Conversely, if $\ell(M) < \infty$ then $M \supset M' \supset 0$ can be upgraded to a comp. series, so $\ell(M') < \infty$, $\ell(M/M') < \infty$.

<u>Thm.</u> (Akizuki-Hopkins). R is artinian $\Leftrightarrow$ R is Noetherian and $\dim R = 0$. If so, then R has fin. many primes.

<u>Pf.</u> If $\dim R = 0$, every prime is maximal. If also R is Noetherian, then R has finite length (by exercise 19.4). So R is Artinian.

Conversely, suppose R Artinian. Let $\mathfrak{m}$ be a minimal product of max. ideals of R.....

<u>Cor.</u> R Artinian, M f.gen. module. Then M has finite length and $\mathrm{Ass}\, M$ and $\mathrm{Supp}\, M$ are equal and finite.

<u>Pf.</u> Every prime is max, so $\mathrm{Supp} M$ consists of max ideals. Also R is Noeth $\Rightarrow$ Exer 19.4 yields the assertions.

<u>Ex 19.4.</u> R Noetherian, M f.g. TFAE:

(1) M has finite length.
(2) $\mathrm{Supp} M$ consists of max. ideals
(3) $\mathrm{Ass}\, M$ consists of max. ideals.

If so, $\mathrm{Ass}(M) = \mathrm{Supp}\, M$ and they are finite.

<u>Pf.</u> $(1) \Rightarrow (2)$ follows from JH thm.
$(2) \Rightarrow (1)$. Have a filtr. with quots $R/\mathfrak{p}_i$. $\mathrm{Ass} M = \{\mathfrak{p}_1, \ldots$
$(2) \Rightarrow (3)$ obvious as $\mathrm{Ass}\, M \subset \mathrm{Supp}\, M$.
$(3) \Rightarrow (2)$. $\mathrm{Supp} M = \bigcup_{q \in \mathrm{Ass}(M)} V(q) = \bigcup_{q \in \mathrm{Ass} M} q = \mathrm{Ass} M$.
and both are finite. Thus (2) holds.

<u>Cor.</u> R is Artinian $\Leftrightarrow$ $\mathrm{length}(R) < \infty$.

<u>Pf.</u> Take $M = R$ $\cancel{\Rightarrow}$ $\ell(R) < \infty \Rightarrow R$ is an Artinian R-module, ~~by previous~~ i.e. an Artinian ring. Conv, if R is Artinian $\Rightarrow R$ finite length by prev. proposition.

Cor. R is Artinian $\Longleftrightarrow$ R a finite product of Artinian local rings; if so then $R = \prod_{m \in \mathrm{Spec}(R)} R_m$.

Pf. A finite product is Artinian $\Leftrightarrow$ each ring is Artinian. (proved before: $M_1, M_2, \ldots, M_r$ Art $\Leftrightarrow$ $M_1 \oplus \cdots \oplus M_r$ Art).
If R Art $\Rightarrow$ $\ell(R) < \infty$, so $R = \prod R_m$ by JH theorem. $\square$