Lecture 19: Length

Let $R$ be a ring, $M$ an $R$-module.

**Def.** $M$ is simple if $M \neq 0$ and its only proper submodule is 0.

A chain $M_0 = M, 0 \hookrightarrow M_1, \ldots \hookrightarrow M_n = 0$ is a composition series of length $m$ if $M_{i-1}/M_i$ are simple.

**Length of $M$** $l(M) = \inf \{ m \mid M \text{ has a comp. series of length } m \}$.

If $M$ has no comp. series, then $l(M) = \infty$. Also $l(M) = 0 \iff M = 0$.

**Ex.** If $R$ is a field, then $l(M) = \dim M$.

**Thm.** (Jordan-Holder). Suppose $M$ has a comp. series $M = M_0 \supset M_1 \supset \cdots \supset M_m = 0$. Then any chain of submodules can be refined to a comp. series and every composition series has the same length $l(M)$.

Also $\text{Supp} M = \{ m \in \text{Spec} R \mid m = \text{Ann} (M_{i-1}/M_i) \}$.

The $m \in \text{Supp} M$ are maximal for some $i$.

There is a canonical isomorphism

$M \cong \prod_{m \in \text{Supp} M} M_m$

and $l(M_m)$ is the number of $i$ with $m = \text{Ann} (M_{i-1}/M_i)$.
Let $M'$ be a proper submodule. Let us show $e(M') < e(M)$.

To do so, let $M'_i = M_i \cap M'_i$.

Then $M'_i \cap M_i = M'_i$. So $M'_i / M'_i \subseteq M_i / M_i$.

Since $M_i / M_i$ is simple, either $M'_i / M'_i = 0$ or $M'_i / M'_i = M_i / M_i$, so

$M'_i + M_i = M_i$. If this holds and $M_i \subseteq M'$ then $M'_i \subseteq M'$. So this can't hold for all $i$, otherwise $M = M'$.

So for some $i$, $M'_i = M_i$, and we can omit one term. So get a chain

for comp. series for $M'$ shorter than that for $M$. Can choose shortest for $M$, so get $e(M') < e(M)$.

Now for any chain $N_0 \supseteq N_1 \supseteq \ldots \supseteq N_n = 0$ let us show that $N_i \leq e(M)$ by induction in $e(M)$. If $e(M) = 0$, it's clear. Assume $e(M) > 1$.

If $n = 0$, clear. If $n \geq 1$ then $e(N_1) < e(M)$ so $n - 1 \leq e(N_1)$ by induction. Ass. $e(M) > 1$.

If $N_{i-1}/N_i$ is not simple, then $3 \leq N_{i-1} \supseteq N'_i \supseteq N_i$, and we can make the
chain longer. Repeating, we can refine the chain to a comp. series in at most \( l(M) - n \) steps.

Supp. a given chain is a comp. series, then \( l(M) \leq n \). But also \( l(M) \geq n \), so \( l(M) = n \). The first assertion is proved.

To proceed, fix a prime \( \rho \). Exactness of localization yields a chain

\[
M_0 = (M_0)_{\rho} \supset (M_1)_{\rho} \supset \ldots \supset (M_n)_{\rho} = 0.
\]

Now consider a max ideal \( m \). If \( \rho = m \) then \((R/m)_{\rho} = R/m\). If \( \rho \neq m \), then \( T_S \subseteq B_{\rho} \), so \((R/m)_{\rho} = 0\). Set \( m_i = \text{Ann}(M_{i-1}/M_i)\), so \( M_{i-1}/M_i = R/m_i \), and \( m_i \) is max as \( M_{i-1}/M_i \) is simple.

Thus \( \text{Supp } M = \{m_1, \ldots, m_n\} \) as \((M_{i-1})_{\rho}/(M_i)_{\rho} = S_0\), \( S_0 \neq m_i \).

If we omit duplicates we get comp. series from \((M_i)_{\rho}\) with \( M_{i-1}/M_i = R/s_0 \) \( (s_0 \neq m_j \text{ for some } j) \). Thus the number of such \( i \) is \( l(M_{\rho}) \).
Finally, consider the canonical map
\[ \varphi: M \to \prod_{m \in \text{Supp}(M)} M_m. \]
To prove that \( \varphi \) is an isomorphism, it suffices to prove that \( \varphi_p \) is an isomorphism for each \( p \). (Check exactness at each maximal ideal.)

Now, localize commutes with finite products. So
\[ \varphi_p: M_p \to \left( \prod_{m \in \text{Supp}(M)} M_m \right)_p = \prod_{m \in \text{Supp}(M)} (M_m)_p = M_p. \]
Thus \( \varphi_p = 1 \) \( \Rightarrow \) \( \text{B.} \)

Cor. \( M \) both artinian and noetherian \( \Rightarrow \) \( M \) has finite length.

Pf. Any chain \( M = N_0 \supseteq \cdots \supseteq N_n = 0 \) has length
\[ m \leq l(M) \] by the theorem. So if \( l(M) < \infty \)
then \( M \) sat both acc and dcc.

Conversely, assume \( M \) is both noetherian and artinian. Form a chain as follows. Set \( M_0 = M \). For \( i \geq 1 \),
if \( M_{i-1} \neq 0 \), take \( \max M_i = M_i - 1 \)
(exists by maxc). By the dcc, the chain terminates. Then the chain is a complete series.
Thm. (Additivity of Length). \( M \supset M' \Rightarrow \ell(M) = \ell(M') + \ell(M/M') \).

Pf. Suppose any compr. sel. for \( M' \) and \( M/M' \) gives rise to a compr. series of \( M \) through \( M' \). So \( \ell(M') < \infty \), \( \ell(M/M') < \infty \), and \( \ell(M) = \ell(M') + \ell(M/M') \). Conversely, if \( \ell(M) < \infty \) then \( M \supset M' \supset 0 \) can be upgraded to a compr. series, so \( \ell(M') < \infty \), \( \ell(M/M') < \infty \).

Thm. (Akizuki-Kokusan). \( R \) is artinian \( \Rightarrow \) \( R \) is Noetherian and \( \dim R = 0 \).
If so, then \( R \) has fin. many primes.

Pf. If \( \dim R = 0 \), every prime is maximal. If also \( R \) is Noetherian, then \( R \) has finite length (by exercise 19.4). So \( R \) is Artinian.

Conversely, suppose \( R \) Artinian. Let \( M \) be a minimal product of max. ideals of \( R \).
Cor. R Artinian, M f.g. module. Then M has finite length and Ass M and Supp M are equal and finite.

Pf. Every prime is max, so Supp M consists of max ideals. Also R is Noetherian ⇒ Exer 19.4 yields the assertions.

Ex 19.4. R Noetherian, M f.g. TFAE:
(1) M has finite length.
(2) Supp M consists of max ideals.
(3) Ass M consists of max ideals.
If so, \( \text{Ass}(M) = \text{Supp } M \) and they are finite.

Pf. (1) ⇒ (2) follows from TH 19.3. Ass M = \( \text{Supp } M \) & obvious as Ass M ⊆ Supp M.
(2) ⇒ (3) \( \text{Supp } M = U V(q) = U q = \text{Ass } M \) \( q \in \text{Ass}(M) \) q ∈ Ass M
and both are finite. Thus (2) holds.

Cor. R is Artinian \( \iff \text{length } (R) < \infty \).
Pf. Take \( M = R \) \( \iff (R) < \infty \) ⇒ R is an Artinian R-module by prev. proposition, i.e., an Artinian ring. Conversely, if R is Artinian ⇒ R finite length by prev. proposition.
Cor. \( R \) is Artinian \( \iff \) \( R \) a finite product of Artinian local rings; if so then \( R = \prod_{m \in \text{Spec}(R)} R_m \).

Pf. A finite product is Artinian \( \iff \) each ring is Artinian. (proved before: \( M_1, M_2, \ldots, M_r \) Art \( \iff \) \( M_1 \oplus \cdots \oplus M_r \) Art).

If \( R \) Art \( \Rightarrow \) \( l(R) < \infty \), so \( R = \prod R_m \) by JH theorem. \( \Box \).