Lecture 18. Primary decomposition.

Def. R a ring, M a module, Q a submodule. If $\operatorname{Ass} M/Q$ consists of a single prime $\mathfrak{p}$, say Q is primary or $\mathfrak{p}$-primary for M.

Ex. $\mathfrak{p} \subset R$ is $\mathfrak{p}$-primary, as $\operatorname{Ass}(R/\mathfrak{p}) = \mathfrak{p}$.

Prop. R Noeth, M f.g. modules, Q a submod. If Q is $\mathfrak{p}$-primary then $\mathfrak{p} = \operatorname{nil}(M/Q)$.

Pf. We know that $\operatorname{nil}(M/Q) = \bigcap\limits_{\mathfrak{q} \in \operatorname{Ass} M/Q} \mathfrak{q}$ $\square$.

Thm. R Noetherian, $M \neq 0$ f. generated mod, Q a submod. Set $\mathfrak{p} = \operatorname{nil}(M/Q)$.
Then TFAE:

(1) $\mathfrak{p}$ prime and Q is $\mathfrak{p}$-primary

(2) $\mathfrak{p} = \operatorname{zdiv}(M/Q)$

(3) $\forall x \in R, m \in M$ s.t. $xm \in Q$ but $m \notin Q$ we have $x \in \mathfrak{p}$.

Pf. Recall $\mathfrak{p} = \bigcap\limits_{\mathfrak{q} \in \operatorname{Ass}(M/Q)} \mathfrak{q}$ and $\operatorname{zdiv} M/Q = \bigcup\limits_{\mathfrak{q} \in \operatorname{Ass} M/Q} \mathfrak{q}$

Thus $\mathfrak{p} \subset \operatorname{zdiv} M/Q$. Also

$(1) \Rightarrow \operatorname{Ass} M/Q = \mathfrak{p} \Rightarrow (2)$.

Conversely, if $x \in \mathfrak{q} \in \operatorname{Ass} M/Q$, but $x \notin \mathfrak{q}' \in \operatorname{Ass} M/Q$, then $x \notin \mathfrak{p}$, but $x \in \operatorname{zdiv} M/Q$, so $(2) \Rightarrow (1)$, e.g. $(1) \Leftrightarrow (2)$

Clearly, (3) means every zero div. of M/Q is nilpotent, or $\mathfrak{p} \supset \operatorname{zdiv}(M/Q)$. But the opposite always holds. Thus $(2) \Leftrightarrow (3)$.

Cor. R Noeth., $\mathfrak{q} \subset R$ proper ideal.
Let $\mathfrak{p} = \sqrt{\mathfrak{q}}$. Then $\mathfrak{q}$ is primary in R iff $\forall x, y \in R$, $xy \in \mathfrak{q}$ but $x \notin \mathfrak{q}$, one necessarily has $y \in \mathfrak{p}$. Then $\mathfrak{p}$ is prime and $\mathfrak{q}$ is $\mathfrak{p}$-primary.

Pf. Clearly, $\mathfrak{q} = \mathrm{Ann}(R/\mathfrak{q})$. So $\mathfrak{p} = \mathrm{nil}(R/\mathfrak{q})$. Thus the result follows from the above.

Prop. R a Noeth. ring, M a f.gen. module, Q a submodule. Set $\mathfrak{p} = \mathrm{nil}(M/Q)$. If $\mathfrak{p}$ is maximal, Q is $\mathfrak{p}$-primary.

Pf. Since $\mathfrak{p} = \bigcap_{\mathfrak{q} \in \mathrm{Ass}(M/Q)} \mathfrak{q}$, if $\mathfrak{p}$ is maximal, then $\mathfrak{p} = \mathfrak{q}$ $\forall \mathfrak{q} \in \mathrm{Ass}(M/Q)$, so $\mathrm{Ass}(M/Q) = \mathfrak{p}$, as desired.

Cor. R a Noetherian ring, $\mathfrak{q}$ an ideal. Set $\mathfrak{p} = \sqrt{\mathfrak{q}}$. If $\mathfrak{p}$ is maximal then $\mathfrak{q}$ is $\mathfrak{p}$-primary.

Proof: Since $\mathfrak{p} = \mathrm{nil}(R/\mathfrak{q})$, this follows from the prev. proposition.

Cor. R Noetherian, $\mathfrak{m} \subset R$ max ideal. An ideal $\mathfrak{q}$ is $\mathfrak{m}$-primary $\iff \exists n\ \mathfrak{m}^n \subset \mathfrak{q} \subset \mathfrak{m}$.

Pf. The cond $m^n \subset q \subset m$ just means that $m = \sqrt{q}$. $\square$

Lemma. R a Noetherian ring, $\mathfrak{p} \subset R$ prime ideal, M a module. Let Q_1, Q_2 be $\mathfrak{p}$-primary submodules, and $Q = Q_1 \cap Q_2$. Then Q is $\mathfrak{p}$-primary.

Pf. Consider the canonical map $M \to M/Q_1 \oplus M/Q_2$. Its kernel is Q, so we have an injection $M/Q \hookrightarrow M/Q_1 \oplus M/Q_2$.

So we have

$$\emptyset \neq \mathrm{Ass}(M/Q) \subset \mathrm{Ass}(M/Q_1) \cup \mathrm{Ass}(M/Q_2).$$

Since both on the right are $\{\mathfrak{p}\}$, we get $\mathrm{Ass}(M/Q) = \{\mathfrak{p}\}$, as desired.

Primary decomposition. R a ring, $M \supset N$ R-module. A primary decomposition of N is a decomp. $N = Q_1 \cap \cdots \cap Q_r$ with the Q_i primary.

Ex. $R = \mathbb{Z}$, $\mathfrak{a} = (n)$, Assoc. primes are prime factors of n. Primary ideals are (p^k). So primary dec. of (n) is $(n) = (p_1^{\alpha_1}) \cap (p_2^{\alpha_2}) \cap \cdots \cap (p_k^{\alpha_k})$ (corr. to prime factorization)

We call the primary decomp. <u>irredundant</u> or <u>minimal</u> if

(1) $N \neq \bigcap_{j \neq i} Q_j$, $\forall i$

(2) Say Q_i are $\mathfrak{p}_i$-primary. Then $\mathfrak{p}_1, \dots, \mathfrak{p}_r$ are distinct.

If so, we call Q_i the $\mathfrak{p}_i$-primary component of the decomp.

If R is Noetherian any primary decomp can be made irredundant by intersecting all the primary submodules with the same prime, and discarding those that are not needed.

<u>Ex.</u> $R = \mathbb{C}[x,y]$, $\mathfrak{a} = \langle x^2, xy \rangle$

Then $\mathfrak{a} = \langle x \rangle \cap \langle x^2, xy, y^n \rangle$

$= \langle x \rangle \cap \langle x^2, y \rangle$

And $\langle x^2, xy, y^n \rangle$ is $\langle x, y \rangle$-primary

So we have ∞ many minimal primary decomp.

<u>Lemma:</u> R a ring, M a module, $N = Q_1 \cap \dots \cap Q_r$ a primary dec. in M. Say Q_i is $\mathfrak{p}_i$-primary. Then

$\mathrm{Ass}(M/N) \subseteq \{\mathfrak{p}_1, \dots, \mathfrak{p}_r\}$

If equality holds and $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ are distinct, then the dec. is irredundant; the converse holds in R is Noetherian.

Pf. Since $N = \bigcap Q_i$, we have an injection $M/N \hookrightarrow M/Q_1 \oplus \cdots \oplus M/Q_r$. So we get $\mathrm{Ass}(M/N) \subseteq \bigcup_i \mathrm{Ass}(M/Q_i)$. This proves the first statement.

If $N = Q_2 \cap \cdots \cap Q_r$ then $\mathrm{Ass}(M/N) \subseteq \{\mathfrak{p}_2, \ldots, \mathfrak{p}_r\}$. So if equality holds and the primes are distinct, the decomp. is irredundant. Conversely, assume $N = Q_1 \cap \cdots \cap Q_r$ is irredundant. ~~Then $P_i \cap Q_i = N$ and $P_i/N \neq 0$.~~ Given i, let $P_i = \bigcap_{j \neq i} Q_j$. Then $P_i \cap Q_i = N$ and $P_i/N \neq 0$. Consider the following two canonical inj:

$P_i/N \hookrightarrow M/Q_i$ and $P_i/N \hookrightarrow M/N$.

Assume R is Noetherian. Then $\mathrm{Ass}(P_i/N) \neq \emptyset$ by the above. So the first inj. yields $\mathrm{Ass}\, P_i/N = \{\mathfrak{p}_i\}$. The second inj yields $\mathfrak{p}_i \in \mathrm{Ass}(M/N)$. Thus $\mathrm{Ass}(M/N) \supset \{\mathfrak{p}_1, \ldots, \mathfrak{p}_r\}$, i.e. $\mathrm{Ass}(M/N) = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_r\}$, as desired.

Thm. (First uniqueness). Let R be a Noetherian ring, M a module. Let $N = Q_1 \cap \dots \cap Q_r$ be an irredundant primary dec. of M. Say Q_i is $\mathfrak{p}_i$-primary for $i = 1, \dots, r$. Then $\mathfrak{p}_1, \dots, \mathfrak{p}_r$ are uniquely determined; in fact, they are just the ~~d~~ distinct primes associated to M/N.

Pf. This follows immediately from the above.

Thm. (Lasker–Noether) Over a Noetherian ring, each proper submodule over a f.g. module has an irredundant primary decomp.

Pf. $N \subset M$. Then by above, M/N has f. many distinct associated primes, $\mathfrak{p}_1, \dots, \mathfrak{p}_r$. Then $\forall i$ $\exists$ a $\mathfrak{p}_i$-primary submodule Q_i (as $\forall \Psi \subset \operatorname{Ass} M \Rightarrow \exists N$ $\operatorname{Ass} N = \Psi$, $\operatorname{Ass} M/N = \operatorname{Ass} M \setminus \Psi$). with $\operatorname{Ass}(Q_i/N) = \operatorname{Ass}(M/N) - \mathfrak{p}_i$. Set $P = \bigcap Q_i$. Fix i. Then $P/N \subset Q_i/N$. So $\operatorname{Ass} P/N \subset \operatorname{Ass} Q_i/N$. But i is arbitrary, so $\operatorname{Ass}(P/N) = \emptyset$. So $P/N = 0$ and $P = N$. Now, the dec. $N = \bigcap_i Q_i$ is irredundant (since all assoc. primes are distinct).

Lemma. R Noetherian, S mult. subset, $\mathfrak{p} \subset R$ prime, M a module, Q a $\mathfrak{p}$-primary submodule.

If $S \cap \mathfrak{p} \neq \emptyset$ then $S^{-1}Q = S^{-1}M$ and $Q^S = M$.
If $S \cap \mathfrak{p} = \emptyset$ then $S^{-1}Q$ is $\mathfrak{p}$-primary in $S^{-1}M$, and $Q^S = \varphi_S^{-1}(S^{-1}Q) = Q$.

Pf. Every prime in $S^{-1}R$ is of the form $S^{-1}\mathfrak{q}$, where $\mathfrak{q}$ is a prime in R with $S \cap \mathfrak{q} = \emptyset$. And $S^{-1}\mathfrak{q} \in \mathrm{Ass}(S^{-1}(M/Q)) \iff \mathfrak{q} \in \mathrm{Ass}\, M/Q$, i.e., $\mathfrak{q} = \mathfrak{p}$ (assoc. primes are preserved under localization). But $S^{-1}(M/Q) = S^{-1}M/S^{-1}Q$. So if $S \cap \mathfrak{p} \neq 0$ then $\mathrm{Ass}(S^{-1}M/S^{-1}Q) = \emptyset$, so get $S^{-1}M/S^{-1}Q = 0$, and $S^{-1}M = S^{-1}Q$. Otherwise, if $S \cap \mathfrak{p} = \emptyset$, then $\mathrm{Ass}(S^{-1}M/S^{-1}Q) = S^{-1}\mathfrak{p}$. So $S^{-1}Q$ is $S^{-1}\mathfrak{p}$-primary is $S^{-1}M$.

Finally, $Q^S = \varphi_S^{-1}(S^{-1}Q)$, so if $S^{-1}Q = S^{-1}M$ then $Q^S = M$. Suppose $S \cap \mathfrak{p} = \emptyset$. Given $m \in Q^S$, $\exists s \in S$ with $sm \in Q$. But $s \notin \mathfrak{p}$. Also $\mathfrak{p} = \mathrm{zdiv}(M/Q)$. Thus $m \in Q$. Hence $Q^S \subset Q$. But $Q^S \supset Q$ as $1 \in Q$. So $Q^S = Q$.

Prop. R Noeth, $S \subset R$ mult, M f.g. mod. Let $N = Q_1 \cap \dots \cap Q_r \subset M$ be an irredundant primary dec. Say Q_i is a P_i-primary $\forall i$, and $S \cap \mathfrak{p}_i = \emptyset$ just for $i \leq h$. Then
$$S^{-1}N = S^{-1}Q_1 \cap \dots \cap S^{-1}Q_h \subset M \text{ and } N^S = Q_1 \cap \dots \cap Q_h \subset M.$$
are irredundant primary decompositions.

Pf. $S^{-1}N = S^{-1}Q_1 \cap \dots \cap S^{-1}Q_r$. (localization commutes with intersection).

Also as shown above $S^{-1}Q_i$ is $S^{-1}\mathfrak{p}_i$ primary for $i \le h$, and $S^{-1}Q_i = S^{-1}M$ if $i > h$. So $S^{-1}N = S^{-1}Q_1 \cap \dots \cap S^{-1}Q_h$ is a primary decomp. It is irredundant by the above. Indeed, $\mathrm{Ass}(S^{-1}M/S^{-1}N) = \{S^{-1}\mathfrak{p}_1, \dots, S^{-1}\mathfrak{p}_h)$, and $S^{-1}\mathfrak{p}_1, \dots, S^{-1}\mathfrak{p}_h$ are distinct.

Apply φ_S^{-1} to $S^{-1}N = S^{-1}Q_1 \cap \dots \cap S^{-1}Q_h$.
We have $N^S = Q_1^S \cap \dots \cap Q_h^S$. But we showed that $Q_i^S = Q_i$. So $N^S = Q_1 \cap \dots \cap Q_h$ is a primary dec. It is clearly irredundant.

<u>Thm.</u> (Second uniqueness). R a ring, $M \supset N$ modules, R Noetherian, $M \supset N$ f.gen. Let $\mathfrak{p}$ be a min prime for M/N. Then in any irredundant primary dec. of N in M, the $\mathfrak{p}$-primary component Q is uniquely determined; in fact, $Q = N^S$ where $S = R \setminus \mathfrak{p}$.

<u>Pf.</u> In the prev. prop, take $S = R \setminus \mathfrak{p}$. Then $h = 1$, since $\mathfrak{p}$ is minimal. $\square$

<u>Thm.</u> (Krull intersection). R Noetherian, $\mathfrak{a} \subset R$ an ideal, M f.g. module. Let $N = \bigcap_{n \ge 0} \mathfrak{a}^n M$. Then $\exists\, x \in \mathfrak{a}$ s.t. $(1+x)N = 0$.

<u>Pf.</u> Since M is f.g, N is f.g. So x exists if $N = \mathfrak{a}N$. (proved before: $N = \mathfrak{a}N \iff \exists\, x \in \mathfrak{a}$ $(1+x)N = 0$).

To prove $N \subset \mathfrak{a}N$, let $\mathfrak{a}N = \bigcap Q_i$,

Q_i are $\mathfrak{p}_i$-primary. Fix i. If $\exists\, a \in \mathfrak{a} \setminus \mathfrak{p}_i$,

then $a\underset{\mathfrak{a}N}{\hat{N}} \subset Q_i$ ~~[illegible]~~, so we get

$N \subset Q_i$ (by what we proved before).

If $\mathfrak{a} \subset \mathfrak{p}_i$, $\exists\, n_i$ s.t. $\mathfrak{a}^{n_i} M \subset Q_i$

(as $\mathfrak{p}_i = \mathrm{nil}(M/Q_i) \Rightarrow \exists\, n_i\ \mathfrak{p}_i^{n_i} \supset \mathrm{Ann}(M/Q_i)$).

as $\mathfrak{p}_i$ is f.g. because R is Noetherian.

So again $N \subset Q_i$. So $N \subset \bigcap Q_i = \mathfrak{a}N$.

<u>Example</u>. ~~$R = \mathbb{C}[x,y]$, $M = \mathbb{C}[x,y]/(xy)$.~~

~~$\mathfrak{a} = (x)$. $N = \bigcap \mathfrak{a}^k M$~~

$R = C^\infty(\mathbb{R})$. m = fns vanishing at 0.

Then m is max (kernel of $f \to f(0)$).

So $m^n = \langle x^n \rangle$ consists of f s.t. first $n-1$ deriv. vanish at 0.

So $\bigcap m^n$ = fns all of whose der. vanish at 0

$\exists\, h \neq 0$ like that: $h = \begin{cases} e^{-\frac{1}{x^2}}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

~~But we have~~ This is not a zero div so $(1+g)h \neq 0$ $\forall g \neq -1$ (e.g. for $g \in m$)

<u>Cor.</u> (of Krull int.). If R is Noetherian domain, $\mathfrak{a} \subset R$ ($\mathfrak{a} \neq R$) an ideal, then $\bigcap \mathfrak{a}^n = 0$.

<u>Pf.</u> $M = R$, get from thm that $\exists\, x \in \mathfrak{a}$ s.t. $1 + x =$

[illegible] $x = -1$