Def. \( R \) a ring, \( M \) a module. \( \mathfrak{p} \subset R \) prime associated to \( M \) if \( \exists m \in M \) s.t. \( \mathfrak{p} = \text{Ann}(m) \). The set of associated primes to \( M \) is denoted by \( \text{Ass} M \). The primes that are minimal in \( \text{Ass} M \) are called \underline{minimal primes of} \( M \). The others are called \underline{embedded primes}.

Rem. Associated primes of an ideal are, by convention, associated primes of \( R/\mathfrak{p} \).

Lemma. \( \mathfrak{p} \) prime, \( M \) an \( R \)-module. Then \( \mathfrak{p} \in \text{Ass}(M) \iff \exists R \text{-inj} \ R/\mathfrak{p} \to M \).

Proof. \( \Rightarrow \). Obvious.

Proof. \( \text{Ass}(M) \subset \text{Supp} M \).

Proof. Let \( \mathfrak{p} \in \text{Ass} M \). Say \( \mathfrak{p} = \text{Ann}(m) \).

Then \( \frac{m}{1} \in M_\mathfrak{p} \) is nonzero, as no \( x \in R - \mathfrak{p} \) sat \( xm = 0 \). So \( M_\mathfrak{p} \neq 0 \) and \( \mathfrak{p} \in \text{Supp} M \).
Lemma. \( R \) \( \neq 0 \) prime, \( m \in R/\mathfrak{a} \) nonzero element. Then \( \text{Ann}(m) = \mathfrak{a} \) and \( \text{Ass}(R/\mathfrak{a}) = \mathfrak{a} \).

**Proof.** Clear.

**Prop.** \( M \supset N \) \( R \)-modules.
Then \( \text{Ass}(N) \subset \text{Ass}(M) \subset \text{Ass}(N) \cup \text{Ass}(M/N) \).

**Proof.** First inclusion is clear. Let \( \mathfrak{a} \in \text{Ass}(M) \)
\( \mathfrak{a} = \text{Ann}(m) \). Then have \( R/\mathfrak{a} \subset M \).
Let \( E \) be the image. If \( E \cap N = 0 \) then \( R/\mathfrak{a} \cap M/N = 0 \) \( \mathfrak{a} \in \text{Ann}(M/N) \).
If \( E \cap N \neq 0 \) then \( \exists n \neq 0 \), \( n \in E \), \( \mathfrak{a} \in \text{Ann}(n) = \mathfrak{a} \) and \( \mathfrak{a} \in \text{Ass}(N) \).

**Prop.** \( M \) an \( R \)-module, \( \Psi \subset \text{Ass}(M) \) a subset.
Then \( E \) a submodule \( N \) of \( M \) with \( \text{Ass}(M/N) = \Psi \) and \( \text{Ass} N = \text{Ass } M - \Psi \).

**Proof.** Given \( N \neq CM \) totally ordered by inclusion, let \( N = \bigcup N_\alpha \). Given \( \mathfrak{a} \in \text{Ass N} \), say \( \mathfrak{a} = \text{Ann}(m) \). Then \( m \in N_\alpha \) for some \( \alpha \), so \( \mathfrak{a} \in \text{Ass } N_\alpha \), so \( \text{Ass } N = \bigcup \text{Ass } N_\alpha \).

So we may obtain using Zorn's lemma
a submodule \( N \subseteq M \) maximal for the property that \( \text{Ass } N \subseteq \text{Ass } M \setminus \gamma. \)

By the above, it suffices to show that \( \text{Ass } (M/N) = \gamma. \) Take \( \gamma \in \text{Ass } (M/N). \)

Then \( M/N \) has a submodule \( N' \) isomorphic to \( R/\gamma. \) So \( \text{Ass } (N') \subseteq \text{Ass } N \cup \{ \gamma \}. \)

By the above. Now, \( N' \nsubseteq N, \) and \( N \) is max for \( \text{Ass } (N) \subseteq \text{Ass } M \setminus \gamma, \) \( \Rightarrow \) \( \text{Ass } N' \nsubseteq \text{Ass } N \setminus \gamma \Rightarrow \gamma \notin \text{Ass } (N \setminus \gamma) \).

\( \Rightarrow \gamma \in \gamma. \)

Prop. \( R = S \) mult. sub. \( M \) an \( R \)-mod., \( \gamma \) CR prime. If \( \gamma \nmid S = \emptyset, \gamma \in \text{Ass } (M) \) then \( \gamma \in \text{Ass } (S'/M); \) the converse holds if \( \gamma \) is f. \( \gamma. \)

Pf. Assume \( \gamma \notin \text{Ass } (M). \) Then have \( R/\gamma \nsubseteq M. \)

This induces an injection \( S'/R/\gamma \simeq S'/M \).

But \( S'(R/\gamma) = S'/R/\gamma \cdot \gamma. \) Assume \( \gamma \notin \text{Ass } (S'). \)

\( \Rightarrow S' \) is a prime. \( \Rightarrow \gamma \in \text{Ass } (S'). \)

Conversely, assume \( \gamma \notin \text{Ass } (S'/M). \) Then \( \exists m \in M, t \in S \) with \( S'/\gamma = \text{Ann } (m/\gamma). \) Say \( \gamma = (\alpha_1, \ldots, \alpha_n) \). Fix \( i. \) Then \( x_i m/\gamma = 0. \) So \( \exists s_i \in S \) s.t. \( S_i x_i m = 0. \) Let \( s = \prod s_i. \) Then \( x_i \in \text{Ann } (s_m). \)

So \( \gamma \notin \text{Ann } (s_m). \)

Let \( b \in \text{Ann } (s_m). \) Then \( b s_m/\gamma = 0. \) So \( b/\gamma \in s_m \gamma. \)

(ce) \( S'/\gamma = \text{Ann } (m/\gamma) \). So \( b \in \gamma. \) Then \( \gamma \notin \text{Ann } (s_m) \Rightarrow \gamma = \text{Ann } (s_m). \)

\( \Rightarrow \) \( \gamma \in \text{Ass } (S). \) Also \( \Rightarrow \gamma \nsubseteq S = \emptyset \) then \( \gamma \notin \text{Ass } (M). \) Since \( S' \) is a prime.
Let $R$ be a ring, $M$ an $R$-module. Then assume $\mathfrak{p}$ is maximal in the set of annihilators of nonzero elements in $M$. Then $\text{PEAnn}(M)$. 

Proof. Say $\mathfrak{p} = \text{Ann}(m)$, $m \neq 0$. Then $1 \notin \mathfrak{p}$. Take $b, c \in R$, $b \in \mathfrak{p}$, but $c \notin \mathfrak{p}$. Then $bcm = 0$ but $cm \neq 0$. Clearly $\mathfrak{p} \subset \text{Ann}(cm)$. So $\mathfrak{p} = \text{Ann}(cm)$ by maximality. But $b \in \text{Ann}(cm)$, so $b \notin \mathfrak{p}$. Hence $\mathfrak{p}$ is a prime.

Prop. $R$ a Noetherian ring, $M$ a module. Then $M = 0 \iff \text{Ass}(M) = \emptyset$.

Proof. Clearly $M = 0 \implies \text{Ass} M = \emptyset$. Conversely, suppose $M \neq 0$, let $S$ be the set of annihilators of nonzero elements of $M$. Then $S$ has a maximal element $\mathfrak{p}$ (any collection of ideals has a maximal element). Then by the above $\mathfrak{p} \in \text{Ass} M$. Thus $\text{Ass} M \neq \emptyset$.

Def. $R$ a ring, $M$ a module, $x \in R$. Say $x$ is a zerodivisor on $M$ if $\exists$ a nonzero $m \in M$ with $xm = 0$; otherwise say $x$ is a nonzerodivisor in $M$. We denote the set of zerodivisors by $\text{2div}(M)$.

Prop. $R$ Noetherian, $M$ a module. Then $\text{2div} M = \bigcup_{\mathfrak{p} \in \text{Ass} M} \mathfrak{p}$. 


Pf. \( \forall x \in \ker M, \text{say } x/m = 0, m \in M, m \neq 0. \)
Then \( x \in \text{Ann}(m). \) But \( \text{Ann}(m) \) is maximal among annihilators of nonzero elements. So \( \mathfrak{p} \in \text{Ass } M. \) So \( \ker(M) \subseteq \mathfrak{p}. \) The opposite inclusion is obvious from def.

Lemma. Let \( R \) be a Noetherian ring, \( M \) an \( R \)-module. Then \( \text{Supp } M = \bigcup \mathcal{V}(\mathfrak{p}) \subseteq \text{Ass } M. \)

Pf. Let \( \mathfrak{p} \) be a prime. Then \( R_{\mathfrak{p}} \) is Noetherian. So \( M_{\mathfrak{p}} \neq 0 \) iff \( \text{Ass } R_{\mathfrak{p}}(M_{\mathfrak{p}}) \neq 0. \) But \( R \) is Noetherian, so \( \text{Ass } R_{\mathfrak{p}}(M_{\mathfrak{p}}) \neq 0 \) (i.e., \( q \in \text{Ass } M \) with \( q \cap (R_{\mathfrak{p}}) \neq 0 \)) by the above. Thus \( \mathfrak{p} \in \text{Supp } M \) (\( \Rightarrow q \in \mathcal{V}(\mathfrak{p}) \)) for some \( q \in \text{Ass } M. \)

Thm. \( R \) a Noetherian ring, \( M \) a module, \( \mathfrak{p} \subseteq \text{Supp } M. \) Then \( \mathfrak{p} \) contains some \( q \in \text{Ass } M. \) If \( \mathfrak{p} \) is minimal is \( \text{Supp } M \), then \( \mathfrak{p} \subseteq \text{Ass } M. \)

Pf. By the above, \( q \) exists. Also \( q \subseteq \text{Supp } M \) so \( \mathfrak{p} = q \) if \( \mathfrak{p} \) is minimal.

Thm. \( R \) a Noetherian ring, \( M \) a f.g. module. Then \( \text{nil}(M) = \bigcap_{\mathfrak{p} \in \text{Ass } M} \mathfrak{p}. \)

Pf. \( M \) f.g. \( \Rightarrow \text{nil}(M) = \bigcap \text{Supp } M, \) as we proved before.
Since $R$ is Noetherian, given $P \in \text{Supp} M$, there is $q \in \text{Ass } M$ with $q \subseteq P$ by the above. The assertion follows.

Let $R$ be a Noetherian ring, $M$ f.g. $R$-mod.

Then there exists a chain of submodules

$$0 = M_0 \subseteq M_1 \subseteq \ldots \subseteq M_n = M$$

with $M_i / M_{i-1} = R/P_i$ for some prime $P_i$, $i = 1, \ldots, n$. For any such chain,

$$\text{Ass } (M) \subseteq \{P_1, P_2, \ldots, P_n\} \subseteq \text{Supp } M.$$ 

Can among $M \neq 0$. Among all submodules having such a chain, there is a max. one $N$ (since such submodules exist, as $\text{Ass } M \neq 0$).

$\text{Supp } M/N \neq 0$. Then $M/N$ and $N/N$ are isomorphic to $R/P_i$ for some prime $P_i$. Then $N \subseteq N$ contradicting maximality. Hence $M = N$.

Thus such a chain exists.

The first inclusion follows by induction from $\text{Ass } M \subseteq \text{Ass } N \cup \text{Ass } (M/N)$.

But $P_i \in \text{Supp } (R/P_i)$, so get the second inclusion. 

Thm. $R$ a Noetherian ring, $M$ f.g. module. Then $\text{Ass } M$ is finite.

Proof. Follows from the above.
Prop. \( M, N \text{ f.g. mod } R \text{ Noetherian.}\)

Then \( \text{Ass } (\text{Hom}(M, N)) = \text{Supp } M \cap \text{Ass } N. \)

Proof. Take \( \mathfrak{p} \in \text{Ass } (\text{Hom}(M, N)). \) Then have \( R/\mathfrak{p} \cong \text{Hom}(M, N). \) Let \( k(\mathfrak{p}) = \text{Frac } (R/\mathfrak{p}). \) Then \( k(\mathfrak{p}) = (R/\mathfrak{p})_0. \) Now, \( M \) is f.p. res.

so \( \text{Hom}(M, N)_{k(\mathfrak{p})} = \text{Hom}_{R_{k(\mathfrak{p})}}(M_{k(\mathfrak{p})}, N_{k(\mathfrak{p})}). \)

so we get \( \psi : k(\mathfrak{p}) \rightarrow \text{Hom}_{R_{k(\mathfrak{p})}}(M_{k(\mathfrak{p})}, N_{k(\mathfrak{p})}). \)

Thus \( M_{k(\mathfrak{p})} \neq 0, \) and \( \mathfrak{p} \in \text{Supp } M. \)

\( \forall \mathfrak{m} \in M_{k(\mathfrak{p})} \text{ with } \mathfrak{p} \mathfrak{m} = 0, \) the map

\( k(\mathfrak{p}) / \mathfrak{m} \rightarrow N_{k(\mathfrak{p})} \)

given by \( x \rightarrow \psi(x)(m). \) is nonzero, so an injection. But \( k(\mathfrak{p}) = R_{k(\mathfrak{p})}/\mathfrak{m} R_{k(\mathfrak{p})}. \)

So \( \mathfrak{p} \in \text{Ass } (N) \) by the above. So \( \mathfrak{p} \in \text{Ass } (M, N) \)

Conversely, take \( \mathfrak{p} \in \text{Supp } M \cap \text{Supp } M \cap \text{Ass } N. \)

Then \( M_{k(\mathfrak{p})} \neq 0. \) So by Nakayama,

\( M_{k(\mathfrak{p})}/\mathfrak{m} M_{k(\mathfrak{p})} \) is a nonzero space over \( k(\mathfrak{p}). \)

Take any nonzero linear map \( M_{k(\mathfrak{p})} \rightarrow k(\mathfrak{p}) \)

precede by can. map \( M_{\mathfrak{p}} \rightarrow M_{k(\mathfrak{p})}/\mathfrak{m} M_{k(\mathfrak{p})}, \)

follow by \( R \)-inj \( k(\mathfrak{p}) \rightarrow N_{k(\mathfrak{p})}. \)

The latter exists since \( \mathfrak{p} \in \text{Ass } N. \)

We obtain a nonzero elt of \( \text{Hom}_{R_{k(\mathfrak{p})}}(M_{k(\mathfrak{p})}, N_{k(\mathfrak{p})}), \) killed by \( \mathfrak{p} R_{k(\mathfrak{p})}. \)
But $\mathfrak{p}\mathfrak{r}$ is maximal, so the whole annihilator. So $\mathfrak{p}\mathfrak{r}\mathcal{C}\mathcal{A}\mathcal{S}\mathcal{S}\mathcal{H}\mathcal{O}\mathcal{M}\mathfrak{r}$ $(M, N)$. Hence $\mathfrak{p}\mathfrak{r}\mathcal{C}\mathcal{A}\mathcal{S}\mathcal{H}\mathcal{O}\mathcal{M} (M, N)$ (as $\text{Hom}_{R_{\mathfrak{p}}} (M, N) = \text{Hom}_{R} (M, N)_{\mathfrak{p}}$ and by the above).

Prop. $R$ Noetherian, $\mathfrak{p}$ a prime, $M$ a f.g. module, $x, y \in \mathfrak{m}$ non-zero divisors on $M$. Then $\mathfrak{p} \in \mathcal{A}\mathcal{S}\mathcal{S} (M/\mathfrak{m}M) \iff \mathfrak{p} \in \mathcal{A}\mathcal{S}\mathcal{S} (M/\mathfrak{y}M)$.

Proof. Have $0 \to K \to M/\mathfrak{m}M \xrightarrow{\mu_y} M/\mathfrak{m}M$, $\nu = \ker (\mu_y)$. Apply $\text{Hom}_{R} (R/\mathfrak{p}, K)$ to this.

Get $0 \to \text{Hom}_{R} (R/\mathfrak{p}, K) \to \text{Hom}_{R} (R/\mathfrak{p}, M/\mathfrak{m}M) \to \ker (\text{Hom}_{R} (R/\mathfrak{p}, M/\mathfrak{m}M))$.

It's exact but $y \in \mathfrak{m}$, so right map vanishes. Thus, $\text{Hom}_{R} (R/\mathfrak{p}, K) \cong \text{Hom}_{R} (R/\mathfrak{p}, M/\mathfrak{m}M)$.

Now form the pull comm. diagram:

$$
\begin{array}{ccc}
0 & \to & M \\
\downarrow & & \downarrow \\
M & \to & M/\mathfrak{m}M \\
\downarrow & & \downarrow \\
K & \to & M/\mathfrak{y}M
\end{array}
$$

Snake lemma yields an ex seq.

$0 \to K \to M/\mathfrak{y}M \xrightarrow{\mu_x} M/\mathfrak{y}M$ as $\ker (\mu_y) = 0$.

Hence $\text{Hom}_{R} (R/\mathfrak{p}, K) \cong \text{Hom}_{R} (R/\mathfrak{p}, M/\mathfrak{y}M)$. So $\text{Hom}_{R} (R/\mathfrak{p}, M/\mathfrak{y}M) = \text{Hom}_{R} (R/\mathfrak{p}, M/\mathfrak{m}M)$. But $\mathfrak{p} \in \mathcal{S}\mathcal{P}\mathcal{P}\mathcal{M}$ $\iff \mathfrak{p} \in \mathcal{A}\mathcal{S}\mathcal{S} (M, N)$ (we are done).