Lecture 17 Associated primes.

<u>Def.</u> R a ring, M a module. $\mathfrak{p} \subset R$ prime associated to M if $\exists\, m \in M$ s.t. $\mathfrak{p} = \mathrm{Ann}(m)$. The set of associated primes to M is denoted by $\mathrm{Ass}\, M$.

The primes that are minimal in $\mathrm{Ass}\, M$ are called <u>minimal primes</u> of M. The others are called <u>embedded primes</u>.

<u>Rem.</u> Associated primes of an ideal are, by convention, associated primes of $R/\mathfrak{a}$.

<u>Lemma.</u> $R \supset \mathfrak{p}$ prime, M an R-module. Then $\mathfrak{p} \in \mathrm{Ass}(M) \iff \exists$ R-inj $R/\mathfrak{p} \hookrightarrow M$.

<u>Pf.</u> Obvious.

<u>Prop.</u> $\mathrm{Ass}(M) \subset \mathrm{Supp}\, M$.

<u>Pf.</u> Let $\mathfrak{p} \in \mathrm{Ass}\, M$. Say $\mathfrak{p} = \mathrm{Ann}(m)$. Then $\frac{m}{1} \in M_{\mathfrak{p}}$ is nonzero, as no $x \in R - \mathfrak{p}$ sat $xm = 0$. So $M_{\mathfrak{p}} \neq 0$ and $\mathfrak{p} \in \mathrm{Supp}\, M$.

<u>Lemma.</u> $R \supset \mathfrak{p}$ prime, $m \in R/\mathfrak{p}$ nonzero element. Then $\mathrm{Ann}(m) = \mathfrak{p}$ and $\mathrm{Ass}(R/\mathfrak{p}) = \mathfrak{p}$.

<u>Pf.</u> Clear.

<u>Prop.</u> $M \supset N$ R-modules.
Then $\mathrm{Ass}(N) \subset \mathrm{Ass}(M) \subset \mathrm{Ass}(N) \cup \mathrm{Ass}(M/N)$.

<u>Pf.</u> First inclusion is clear. Let $\mathfrak{p} \in \mathrm{Ass}(M)$, $\mathfrak{p} = \mathrm{Ann}(m)$. ~~If~~ Then have $R/\mathfrak{p} \subset M$. Let E be the image. If $E \cap N = 0$ then $R/\mathfrak{p} \hookrightarrow M/N \Rightarrow \mathfrak{p} \in \mathrm{Ann}(M/N)$. If $E \cap N \neq 0$ then $\exists n \neq 0$, $n \in E$, $\in N$, so $\mathrm{Ann}(n) = \mathfrak{p}$ and $\mathfrak{p} \in \mathrm{Ass}(N)$.

<u>Prop.</u> M an R-module, $\Psi \subset \mathrm{Ass}(M)$ a subset. Then $\exists$ a submodule N of M with $\mathrm{Ass}(M/N) = \Psi$ and $\mathrm{Ass}\, N = \mathrm{Ass}\, M - \Psi$.

<u>Pf.</u> Given $N_\lambda \subset M$ totally ordered by inclusion, let $N = \bigcup_\lambda N_\lambda$. Given $\mathfrak{p} \in \mathrm{Ass}\, N$, say $\mathfrak{p} = \mathrm{Ann}(m)$. Then $m \in N_\lambda$ for some λ, so $\mathfrak{p} \in \mathrm{Ass}\, N_\lambda$, so $\mathrm{Ass}\, N = \bigcup_\lambda \mathrm{Ass}\, N_\lambda$.
So we may obtain using Zorn's lemma

a ~~sumbmo~~ submodule $N \subset M$ maximal for the property that $\operatorname{Ass} N \subset \operatorname{Ass} M \setminus \Psi$. By the above, it suffices to show that $\operatorname{Ass}(M/N) = \Psi$. Take $\mathfrak{p} \in \operatorname{Ass}(M/N)$. Then M/N has a submodule N'/N isom to $R/\mathfrak{p}$. So $\operatorname{Ass}(N') \subseteq \operatorname{Ass} N \cup \{\mathfrak{p}\}$ by the above. Now, $N' \supsetneq N$, and N is max for $\operatorname{Ass}(N) \subseteq \operatorname{Ass} M \setminus \Psi$, so

$\operatorname{Ass} N' \not\subseteq \operatorname{Ass} N \setminus \Psi \Rightarrow \mathfrak{p} \notin \operatorname{Ass}(N) \setminus \Psi$.

$\Rightarrow \mathfrak{p} \in \Psi$.

__Prop.__ $R \supset S$ mult. subset, M an R-mod, $\mathfrak{p} \subset R$ prime. If $\mathfrak{p} \cap S = \emptyset$, $\mathfrak{p} \in \operatorname{Ass}(M)$ then $S^{-1}\mathfrak{p} \in \operatorname{Ass}(S^{-1}M)$; the converse holds if $\mathfrak{p}$ is f.gen.

__Pf.__ Assume $\mathfrak{p} \in \operatorname{Ass}(M)$. Then have $R/\mathfrak{p} \hookrightarrow M$. This induces an injection $S^{-1}(R/\mathfrak{p}) \hookrightarrow S^{-1}M$. Assume $\mathfrak{p} \cap S = \emptyset$. Then

But $S^{-1}(R/\mathfrak{p}) = S^{-1}R/S^{-1}\mathfrak{p}$. $\mathfrak{p}S^{-1}R$ is a prime. But $\mathfrak{p}S^{-1}R = S^{-1}\mathfrak{p}$. So $S^{-1}\mathfrak{p} \in \operatorname{Ass}_M$.

Conversely, assume $S^{-1}\mathfrak{p} \in \operatorname{Ass}(S^{-1}M)$. Then $\exists\, m \in M$, $t \in S$ with $S^{-1}\mathfrak{p} = \operatorname{Ann}(m/t)$. Say $\mathfrak{p} = (x_1, \dots, x_n)$. Fix i. Then $x_i m/t = 0$. So $\exists\, s_i \in S$ s.t. $s_i x_i m = 0$. Let $s = \prod_i s_i$. Then $x_i \in \operatorname{Ann}(sm)$. So $\mathfrak{p} \subset \operatorname{Ann}(sm)$.

Let $b \in \operatorname{Ann}(sm)$. Then $bsm/st = 0$. So $b/1 \in S^{-1}\mathfrak{p}$ (as $S^{-1}\mathfrak{p} = \operatorname{Ann}(m/t)$). So $b \in \mathfrak{p}$. Thus $\mathfrak{p} \supset \operatorname{Ann}(sm) \Rightarrow \mathfrak{p} = \operatorname{Ann}(sm)$ so $\mathfrak{p} \in \operatorname{Ass}(M)$. Also ~~#~~ $\mathfrak{p} \cap S = \emptyset$ ~~then and thus~~ since $S^{-1}\mathfrak{p}$ is a prime.

<u>L.</u> $R \supset \mathfrak{p}$, M an R-module. ~~Then~~ Assume $\mathfrak{p}$ is maximal in the set of annihilators of nonzero elements in M. Then $\mathfrak{p} \in \mathrm{Ass}(M)$.

<u>Pf.</u> Say $\mathfrak{p} = \mathrm{Ann}(m)$, $m \neq 0$. Then $1 \notin \mathfrak{p}$. Take $b, c \in R$, $bc \in \mathfrak{p}$, but $c \notin \mathfrak{p}$. Then $bcm = 0$ but $cm \neq 0$. Clearly $\mathfrak{p} \subset \mathrm{Ann}(cm)$. So $\mathfrak{p} = \mathrm{Ann}(cm)$ by maximality. But $b \in \mathrm{Ann}(cm)$, so $b \in \mathfrak{p}$. Hence $\mathfrak{p}$ is a prime.

<u>Prop.</u> R a Noetherian ring, M a module. Then $M = 0 \iff \mathrm{Ass}(M) = \emptyset$.

<u>Pf.</u> Clearly $M = 0 \Rightarrow \mathrm{Ass}\, M = \emptyset$. Conversely, supp. $M \neq \emptyset$. Let S be the set of annih. of nonzero elem. of M. Then S has a max. element $\mathfrak{p}$ (any collection of ideals has a max element). Then by the above $\mathfrak{p} \in \mathrm{Ass}\, M$. Thus $\mathrm{Ass}\, M \neq \emptyset$.

<u>Def.</u> R a ring, M a module, $x \in R$. Say x is a zerodivisor on M if $\exists$ a nonzero $m \in M$ with $xm = 0$; otherwise say x a nonzerodivisor in M. We denote the set of zerodivisors by $\mathrm{zdiv}(M)$.

<u>Prop.</u> R Noetherian, M a module. Then
$$\mathrm{zdiv}\, M = \bigcup_{\mathfrak{p} \in \mathrm{Ass}\, M} \mathfrak{p}.$$

Pf. $\forall x \in \mathrm{zdiv} M$, say $xm = 0$, $m \in M$, $m \neq 0$. Then $x \in \mathrm{Ann}(m)$. But $\mathrm{Ann}(m) \subset \mathfrak{p}$ maximal among annihilators of nonzero elements. So $\mathfrak{p} \in \mathrm{Ass} M$. So $\mathrm{zdiv}(M) \subset \bigcup \mathfrak{p}$. The opposite inclusion is obvious from def.

Lemma. Let R be a Noetherian ring, M a module. Then $\mathrm{Supp} M = \bigcup_{\mathfrak{q} \in \mathrm{Ass} M} V(\mathfrak{q}) \supset \mathrm{Ass} M$.

Pf. Let $\mathfrak{p}$ be a prime. Then $R_\mathfrak{p}$ is Noetherian. So $M_\mathfrak{p} \neq 0$ iff $\mathrm{Ass}_{R_\mathfrak{p}}(M_\mathfrak{p}) \neq 0$. But R is Noeth., so $\mathrm{Ass}_{R_\mathfrak{p}}(M_\mathfrak{p}) \neq 0 \Leftrightarrow \exists \mathfrak{q} \in \mathrm{Ass} M$ with $\mathfrak{q} \cap (R \setminus \mathfrak{p}) \neq 0$ (i.e. $\mathfrak{q} \subset \mathfrak{p}$), by the above. Thus $\mathfrak{p} \in \mathrm{Supp} M$ $\Leftrightarrow \mathfrak{p} \in V(\mathfrak{q})$ for some $\mathfrak{q} \in \mathrm{Ass} M$.

Thm. R Noetherian, M a module, $\mathfrak{p} \subset \mathrm{Supp} M$. Then $\mathfrak{p}$ contains some $\mathfrak{q} \in \mathrm{Ass} M$. If $\mathfrak{p}$ is minimal is $\mathrm{supp} M$ then $\mathfrak{p} \in \mathrm{Ass} M$.

Pf. By the above, $\mathfrak{q}$ exists. Also $\mathfrak{q} \in \mathrm{Supp} M$, so $\mathfrak{p} = \mathfrak{q}$ if $\mathfrak{p}$ is minimal.

Thm. R a Noetherian ring, M a f.g. module. Then $\mathrm{nil}(M) = \bigcap_{\mathfrak{p} \in \mathrm{Ass} M} \mathfrak{p}$.

Pf. M f.gen $\Rightarrow$ $\mathrm{nil}(M) = \bigcap_{\mathfrak{p} \in \mathrm{Supp} M} \mathfrak{p}$, as we proved before.

Since R is Noetherian, given $\mathfrak{p} \in \operatorname{Supp} M$, there is $\mathfrak{q} \in \operatorname{Ass} M$ with $\mathfrak{q} \subset \mathfrak{p}$ by the above. The assertion follows.

<u>L.</u> Let R be a Noetherian ring, M f.g. R-mod. Then $\exists$ a chain of submodules

$$0 = M_0 \subset M_1 \subset \cdots \subset M_n = M$$

with $M_i/M_{i-1} = R/\mathfrak{p}_i$ for some prime $\mathfrak{p}_i$, $i = 1, \dots, n$. For any such chain,

$$\operatorname{Ass}(M) \subset \{\mathfrak{p}_1, \dots, \mathfrak{p}_n\} \subset \operatorname{Supp} M.$$

<u>Pf.</u> Can assume $M \neq 0$. Among all submodules having such a chain, there is a max. one N (since such submodules exist, as $\operatorname{Ass} M \neq 0$.

Supp. $M/N \neq 0$. Then M/N cont N'/N isom. to $R/\mathfrak{p}$ for some prime $\mathfrak{p}$. Then $N \subsetneq N'$, contradicting maximality. Hence $M = N$. Thus such a chain exists.

The first inclusion follows by induction from $\operatorname{Ass} M \subset \operatorname{Ass} N \cup \operatorname{Ass}(M/N)$.

But $\mathfrak{p}_i \in \operatorname{Supp}(R/\mathfrak{p}_i)$, so get the second incl. ⊠

<u>Thm.</u> R a Noetherian ring, M f.g. module. Then $\operatorname{Ass} M$ is finite.

<u>Pf.</u> Follows from the above.

Prop. M, N f.g. mod $/R$ Noetherian.
Then
$\mathrm{Ass}(\mathrm{Hom}(M,N)) = \mathrm{Supp}\, M \cap \mathrm{Ass}\, N$.

Pf. Take $\mathfrak{p} \in \mathrm{Ass}(\mathrm{Hom}(M,N))$. Then have
$R/\mathfrak{p} \hookrightarrow \mathrm{Hom}(M,N)$. Let $k(\mathfrak{p}) = \mathrm{Frac}(R/\mathfrak{p})$.
Then $k(\mathfrak{p}) = (R/\mathfrak{p}R)_{\mathfrak{p}}$. Now, M is f. pres.
so $\mathrm{Hom}(M,N)_{\mathfrak{p}} = \mathrm{Hom}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}, N_{\mathfrak{p}})$.
so we get $\varphi : k(\mathfrak{p}) \hookrightarrow \mathrm{Hom}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}, N_{\mathfrak{p}})$.
Thus $M_{\mathfrak{p}} \neq 0$, and $\mathfrak{p} \in \mathrm{Supp}\, M$.
$\forall\, m \in M_{\mathfrak{p}}$ with $\varphi(1)(m) \neq 0$, the map
$k(\mathfrak{p}) \to N_{\mathfrak{p}}$ given by $x \to \varphi(x)(m)$ is
nonzero, so an injection. But $k(\mathfrak{p}) = R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$
so $\mathfrak{p}R_{\mathfrak{p}} \subset \mathrm{Ass}(N_{\mathfrak{p}})$. Thus we also have
$\mathfrak{p} \in \mathrm{Ass}(N)$ by the above. So $\mathrm{Ass}\,\mathrm{Hom}(M,N) \subseteq \mathrm{Supp} M \cap \mathrm{Ass}\, N$.
Conversely, take $\mathfrak{p} \in \mathrm{Supp} M \cap \mathrm{Ass}\, N$.
Then $M_{\mathfrak{p}} \neq 0$. So by Nakayama,
$M_{\mathfrak{p}}/\mathfrak{p}M_{\mathfrak{p}}$ is a nonzero space over $k(\mathfrak{p})$.
Take any nonzero linear map $M_{\mathfrak{p}}/\mathfrak{p}M_{\mathfrak{p}} \to k(\mathfrak{p})$.
precede by can. map $M_{\mathfrak{p}} \to M_{\mathfrak{p}}/\mathfrak{p}M_{\mathfrak{p}}$,
follow by R-inj $k(\mathfrak{p}) \hookrightarrow N_{\mathfrak{p}}$.
The latter exists since $\mathfrak{p} \in \mathrm{Ass}\, N$.
We obtain a nonzero elt of
$\mathrm{Hom}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}, N_{\mathfrak{p}})$, killed by $\mathfrak{p}R_{\mathfrak{p}}$.

But $\mathfrak{p} R_{\mathfrak{p}}$ is maximal, so the whole annihilator. So $\mathfrak{p} R_{\mathfrak{p}} \subset \operatorname{Ass} \operatorname{Hom}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}, N_{\mathfrak{p}})$.
Hence $\mathfrak{p} \in \operatorname{Ass} \operatorname{Hom}(M,N)$ (as
$\operatorname{Hom}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}, N_{\mathfrak{p}}) = \operatorname{Hom}_R(M,N)_{\mathfrak{p}}$ and by the above).

<u>Prop</u>: R Noetherian, $\mathfrak{p}$ a prime, M a f.g. module, $x, y \in \mathfrak{p}$ nonzerodiv. on M. Then $\mathfrak{p} \in \operatorname{Ass}(M/xM) \iff \mathfrak{p} \in \operatorname{Ass}(M/yM)$.

<u>Pf</u>. Have $0 \to K \to M/xM \xrightarrow{\mu_y} M/xM$, $K = \operatorname{Ker}(\mu_y)$. Apply $\operatorname{Hom}(R/\mathfrak{p}, ?)$ to this. Get
$$0 \to \operatorname{Hom}(R/\mathfrak{p}, K) \to \operatorname{Hom}(R/\mathfrak{p}, M/xM) \to \operatorname{Hom}(R/\mathfrak{p}, M/xM)$$
It's exact. But $y \in \mathfrak{p}$, so right map vanishes. Thus, $\operatorname{Hom}(R/\mathfrak{p}, K) = \operatorname{Hom}(R/\mathfrak{p}, M/xM)$.
Now form the foll. comm. diagram with exact rows:
$$\begin{tikzcd} & & & K \arrow[d, dashed] & \\ 0 \arrow[r] & M \arrow[r, "\mu_x"] \arrow[d, "\mu_y"] & M \arrow[r] \arrow[d, "\mu_y"] & M/xM \arrow[r] \arrow[d, "\mu_y"] & 0 \\ 0 \arrow[r] & M \arrow[r, "\mu_x"] & M \arrow[r] & M/xM \arrow[r] & 0 \end{tikzcd}$$
Snake lemma yields an ex. seq.
$$0 \to K \to M/yM \xrightarrow{\mu_x} M/yM \quad \text{as } \operatorname{Ker} \mu_y|_M = 0.$$
Hence, $\operatorname{Hom}(R/\mathfrak{p}, K) = \operatorname{Hom}(R/\mathfrak{p}, M/yM)$. So $\operatorname{Hom}(R/\mathfrak{p}, M/yM) = \operatorname{Hom}(R/\mathfrak{p}, M/xM)$. But $\mathfrak{p} \in \operatorname{Supp} R/\mathfrak{p}$ and $\operatorname{Ass} \operatorname{Hom}(M,N) = \operatorname{Supp} M \cap \operatorname{Ass} N$, so we are done.