Lecture 15. Noether normalization.

Lemma. (Noether normalization). Let k be a field, $R = k[x_1, .., x_n]$ a f.gen. k-algebra (not nec. free), and $\mathfrak{a}_1 \subset \cdots \subset \mathfrak{a}_r$ a chain of proper ideals of R. Then $\exists$ alg. indep. elements $t_1, .., t_\nu \in R$ s.t.

(1) R is module finite over $P = k[t_1, .., t_\nu]$

(2) $\forall i = 1, .., r$ $\exists h_i$ s.t. $\mathfrak{a}_i \cap P = \langle t_1, .., t_{h_i} \rangle$.

If k is infinite, we may choose t_i to be k-linear combinations of the x_i to satisfy condition (1) only.

Pf. Step 1. Reduction to alg. ind. case.

Let $R' = k[X_1, .., X_n]$, $\varphi : R' \twoheadrightarrow R$.

Let $\mathfrak{a}_0' = \ker \varphi$, $\mathfrak{a}_i' = \varphi^{-1} \mathfrak{a}_i$. It suffices to prove the lemma for R' and $\mathfrak{a}_0' \subset \mathfrak{a}_1' \subset \cdots \subset \mathfrak{a}_r'$: if $t_i' \in R'$ and h_i' work here

$t_i = \varphi t'_{i+h_0'}$ and $h_i = h_i' - h_0'$ works for R and $\mathfrak{a}_i$, and clearly (1) and (2) hold.

Step 2. Induction in R (and $\nu = n$).

First assume $r = 1$ and $\mathfrak{a}_1 = t_1 R$, $t_1 \neq 0$.

Then $t_1 \notin k$ as $\mathfrak{a}_1$ proper. Supp. we have found $t_2, .., t_n \in R$ s.t. x_1 is integral over $P = k[t_1, .., t_r]$ and so that $P[x_1] = R$.

Then get (1). Also $t_1, \dots, t_n$ alg. indep.
Now take $x = \mathfrak{a}_1 \cap P$. Then $x = t_1 x'$,
$x' \in R \cap \mathrm{Frac}(P)$. Also $R \cap \mathrm{Frac}\, P = P$
since P is normal (a pol. alg). So
$\mathfrak{a}_1 \cap P = t_1 P$. This gives (2).

To find $t_2, \dots, t_n$, choose l_i and set
$t_i = x_i - x_1^{l_i}$. Then clearly $P[x_1] = R$.
Now, say, $t_1 = \sum a_{\vec{j}}\, x_1^{j_1} \cdots x_n^{j_n}$
Recall $t_1 \notin k$, and note that x_1 satisfies
$$\sum a_{\vec{j}}\, x_1^{j_1} (t_2 + x_1^{l_2})^{j_2} \cdots (t_n + x_1^{l_n})^{j_n} = t_1.$$
Let $e(\vec{j}) = j_1 + l_2 j_2 + \dots + l_n j_n$. Take $l = \max(j_i)$
$l_i = l^i$. Then $e(j)$ are distinct.
Let $e(\vec{j'})$ be the largest of $e(j)$
with $a(j) \neq 0$. Then $e(j') > 0$,
and can write the above
eqn. as
$$a_{j'} x_1^{e(j')} + \sum_{e < e(\vec{j'})} p_e x_1^e = 0.$$
Thus x_1 is integral over P,
as desired.

Now assume $r=1$ and $\mathfrak{a}_1$ is arbitrary. We may assume $\mathfrak{a}_1 \neq 0$. The proof is by induction in n. The case $n=1$ follows from the first case (easier since $k[x_1]$ is a PID. Let $t_1 \in \mathfrak{a}_1$ be nonzero. By the first case, there are elements $u_2, \dots, u_n$ s.t. $t_1, u_2, \dots, u_n$ are alg. indep. and satisfy (1), (2) for R & ideal $t_1 R$. By ind, there are $t_2, \dots, t_n$ sat. (1), (2) for $k[u_2, \dots, u_n]$ and $\mathfrak{a}_1 \cap k[u_2, \dots, u_n]$. Let $P = k[t_1, \dots, t_n]$. Since R is module finite over $k[t_1, u_2, \dots, u_n]$, and $k[t_1, u_2, \dots, u_n]$ is module finite over P, we get that ~~all~~ R is module finite over P. Thus (1) holds, and $t_1, \dots, t_n$ are alg. independent. Further, by assumption $\mathfrak{a}_1 \cap k[t_2, \dots, t_n] = (t_2, \dots, t_h)$ for some h. But $t_1 \in \mathfrak{a}_1$, so $\mathfrak{a}_1 \cap P \supset \langle t_1, \dots, t_h \rangle$.

To prove the opposite inclusion, let $x \in \mathfrak{a}_1 \cap P$. Say $x = \sum_{i=0}^{d} f_i t_1^i$, $f_i \in k[t_2, \dots, t_n]$.

Since $t_1 \in \mathfrak{a}_1$, we have $f_0 \in \mathfrak{a}_1 \cap k[t_2, .., t_n]$. So $f_0 \in \langle t_2, .., t_n \rangle$. Thus $x \in \langle t_1, .., t_n \rangle$. So $\mathfrak{a}_1 \cap P = \langle t_1, .., t_h \rangle$ and (2) hold, as desired (for $r = 1$).

Finally, assume that the lemma holds for $r-1$. Let $u_1, .., u_n \in R$ be alg. ~~ind~~. elts satisfying (1) and (2) for the sequence $\mathfrak{a}_1 \subset .. \subset \mathfrak{a}_{r-1}$, and set $h = h_{r-1}$. By the second case, $\exists$ $t_{h+1}, .., t_n$ satisfying (1) and (2) for $k[u_{h+1}, .., u_n]$ and $\mathfrak{a}_r \cap k[u_{h+1}, .., u_n]$. Thus, for some h_r, $\mathfrak{a}_r \cap k[t_{h+1}, .., t_n] = \langle t_{h+1}, .., t_{h_r} \rangle$. Set $t_i = u_i$ for $1 \le i \le h$. Let $P = k[t_1, .., t_n]$. Since R is module finite over $k[u_1, .., u_n]$ and $k[u_1, .., u_n]$ module finite over $k[t_1, .., t_n]$, we get that R is module finite over $k[t_1, .., t_n]$. So (1) holds, and $t_1, .., t_n$ are alg. indep/k.

Fix i with $1 \le i \le r$. Let $m = h_i$. Then $t_1, .., t_m \in \mathfrak{a}_i$. Then $t_1, .., t_m \in \mathfrak{a}_i$, so $\mathfrak{a}_i \cap P \supset \langle t_1, .., t_{h_i} \rangle$. To prove the opposite inclusion, let $x \in \mathfrak{a}_i \cap P$, say

$$x = \sum f_v \, t_1^{v_1} \cdots t_m^{v_m}.$$

$f_0 \in k[t_{m+1},\dots,t_n]$. Then f_0 lies in $\mathfrak{a}_i \cap k[t_{m+1},\dots,t_n]$. We will see that the latter intersection equals 0. It is so for $i \leq r-1$, as it lies in $\mathfrak{a}_i \cap k[u_{m+1},\dots,u_n]$, which is 0. Further, if $i=r$, then by assumption, $\mathfrak{a}_i \cap k[t_{m+1},\dots,t_n] = \langle t_{m+1},\dots,t_m\rangle = 0$. empty list

Thus $f_0 = 0$, so $x = \langle t_1,\dots,t_{n_i}\rangle$.

So $\mathfrak{a}_i \cap P \subset \langle t_1,\dots,t_{n_i}\rangle$, hence $\mathfrak{a}_i \cap P = \langle t_1,\dots,t_{n_i}\rangle$. Thus we get (2) and the Noether normalization lemma is proved.

<u>Thm.</u> (Zariski Nullstellensatz).
Let k be a field, $R \supset k$ f.g. algebra. Assume R is a field. Then R/k is finite (as a module).

<u>Pf.</u> By Noether, R is module finite over $P = k[t_1, \dots, t_\nu]$. Then R/P is integral. As R is a field, so is P as shown above. Hence $\nu = 0$. So $P = k$ and R/k is finite.

<u>Cor.</u> Let k be a field, $R = k[x_1, \dots, x_n]$ an algebra-finite extension, $m \subset R$ max. ideal. Assume k alg. closed. Then $\exists\, a_1, \dots, a_n$, $m = (x_1 - a_1, \dots, x_n - a_n)$.

<u>Pf.</u> Set $K = R/m$. Then K is a field ext. of k which is finite by Zariski's Nullstellensatz. But k is alg. cl $\Rightarrow$ $K = k$. Let $a_i \in k$ be the residue of x_i, and $n = (x_1 - a_1, \dots, x_n - a_n)$. Then $n \subset m$. Let $R' = k[X_1, \dots, X_n]$ be the pol. ring, $\varphi : R' \twoheadrightarrow R$. ~~Then~~ $\cancel{R' \twoheadrightarrow R \xrightarrow{\psi} k}$
$n' = (X_1 - a_1, \dots, X_n - a_n)$. So $\varphi(n') = n$.
But n' is max, so n is max. Hence $n = m$.

Cor. Let k be any field, ~~R a f.gen k-algebra~~. $P = k[X_1, \dots, X_n]$ pol. ring, $\mathfrak{m}$ a maximal ideal of P.
Then $\mathfrak{m}$ is generated by n elements.

Pf. Let $K = P/\mathfrak{m}$. Then K is a finite ext of k. Induct on n. If $n = 0$, $\mathfrak{m} = 0$. Assume $n \geq 1$. Let $R = k[X_1]$, $\mathfrak{p} = \mathfrak{m} \cap R$. Then $\mathfrak{p} = (f_1)$ as K is a PID. Let $k_1 = R/\mathfrak{p}$. Then k_1 is the image of R in K. So k_1 is f.d.$/k$, hence integral ext. Thus k_1 is a field. Now $P/\mathfrak{p}P = k_1[X_2, \dots, X_n]$, and $\mathfrak{m}/\mathfrak{p}$ is a max ideal in this ring. So by the ind ass. it is gen. by $n-1$ elts, say residues of $f_2, \dots, f_n \in \mathfrak{m}$. Then $\mathfrak{m} = (f_1, \dots, f_n)$.

Theorem. (Hilbert's Nullstellensatz).
Let k be a field, R a f.g. k-algebra. Let $\mathfrak{a} \subset R$ be a proper ideal.
Then $\sqrt{\mathfrak{a}} = \bigcap_{\mathfrak{m} \supset \mathfrak{a}} \mathfrak{m}$ where $\mathfrak{m}$ runs through all max ideals containing $\mathfrak{a}$.

Pf. We may assume $\mathfrak{a}=0$ by replacing R with $R/\mathfrak{a}$. Clearly, $\sqrt{0} \subset \bigcap m$.
Conversely, let $f \notin \sqrt{0}$. Then $R_f \neq 0$.
So R_f has a max ideal n. Let m be its contraction to R, $m = \varphi^{-1}(n)$.
Note that R_f is f. gen k-alg.
Thus R_f/n is a finite extension of k.
Let $K = R/m$. By construction
$K \subset R_f/n$ a k-subalg. So K finite ext of k, hence a field. Thus m is maximal.
But $\frac{f}{1}$ is a unit in R_f, so $\frac{f}{1} \notin n$. Hence
$f \notin m$. So $f \notin \bigcap$ max ideals. Thus,
$\sqrt{0} = \bigcap m$.

Remark. Supp. k alg. closed.
Then $R = k[X_1, \dots, X_n] / (f_\alpha)$ - system of pol. eqns.
(closed alg. set). So $\underset{\text{max ideals}}{m} \longleftrightarrow$ solutions
X $\qquad$ X - set of sol.

Nullstellensatz say that if f vanishes on X then $f^n \in (f_\alpha)$ for some n.

<u>Lemma.</u> k a field, R f.g. k-algebra. Assume R is a domain. Let $\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_r$ be a chain of primes. Let $K = \mathrm{Frac}(R)$ and $d = \mathrm{trdeg}_k K$. Then $r \leq d$, with equality only if the chain is <u>maximal</u>, i.e. not a proper subchain of a longer chain.

<u>Pf.</u> By the Noether normalization lemma, R is module finite over a polyn. subring $P = k[t_1, \dots, t_\nu]$ s.t. $\mathfrak{p}_i \cap P = \langle t_1, \dots, t_{h_i} \rangle$ for some h_i.
Let $L = \mathrm{Frac}(P)$. Then $\nu = \mathrm{trdeg}\, L$.
But $P \subset R$ is an int. ext. So $L \subset K$ algebraic.
Thus $\nu = d$. Now, incomparability tells us $h_i < h_{i+1}$ for all i, so $r \leq h_r$.
But $h_r \leq \nu$, $\nu = d$, so $r \leq d$.

If $r = d$ then r is maximal. Conversely, assume r is maximal. Then $\mathfrak{p}_0 = \langle 0 \rangle$, as R is a domain. So $h_0 = 0$. Also $\mathfrak{p}_r$ is maximal since $\mathfrak{p}_r$ is contained in some max. ideal, and it is a prime. So $\mathfrak{p}_r \cap P$ is maximal also (ideal over maximal is maximal, and vice versa).

Thus $h_r = r$. Suppose $\exists i$ s.t. $h_i + 1 < h_{i+1}$.

Then $(p_i \cap P) \subsetneq (t_1, \ldots, t_{h_i+1}) \subsetneq P_{i+1} \cap P$.

Now, $P/_{p_i \cap P} = k[t_{h_i+1}, \ldots, t_d]$, and the

ext. $P/_{p_i \cap P} \subseteq R/_{p_i}$ is integral.

So by going down thm, we have a

prime $p_i < p < p_{i+1}$ with $p \cap P = \langle t_1, \ldots, t_{h_i+1} \rangle$

Then we get a longer chain, $\Rightarrow \Leftarrow$.

So $h_{i+1} = h_i + 1$ $\forall i$, thus $r = h_r$, so $r = d$.

<u>Def.</u> <u>Krull dimension</u> $\dim(R)$

is the supremum of lengths r of all

strictly ascending chains of primes.

$p_0 \subsetneq \cdots \subsetneq p_r$

E.g. if R is a field, $\dim R = 0$,

and more generally $\dim R = 0$ if

every prime is maximal. If R is a PID

but not a field, $\dim R = 1$, as any

prime $\neq 0$ is maximal.

<u>Thm.</u> k a field, R f.gen k-algebra. If R is a domain, then $\dim R = \operatorname{trdeg} \operatorname{Frac}(R)$.

<u>Pf.</u> This immediately follows from the above.

<u>Thm.</u> Let k be a field, R a f.gen. k-alg; $\mathfrak{p}$ a prime ideal, $\mathfrak{m}$ a max. ideal. Supp R is a domain. Then

$$\dim R_{\mathfrak{p}} + \dim R/\mathfrak{p} = \dim R$$

and $\dim R_{\mathfrak{m}} = \dim R$.

<u>Pf.</u> A chain of primes $\mathfrak{p}_0 \subsetneq \cdots \subsetneq \mathfrak{p} \subsetneq \cdots \subset \mathfrak{p}_r$ gives rise to a pair of chains of primes, one for $R_{\mathfrak{p}}$ and the other for $R/\mathfrak{p}$:

$$\mathfrak{p}_0 R_{\mathfrak{p}} \subsetneq \cdots \subsetneq \mathfrak{p} R_{\mathfrak{p}} \quad \text{and} \quad 0 = \mathfrak{p}/\mathfrak{p} \subsetneq \cdots \subsetneq \mathfrak{p}_r/\mathfrak{p}.$$

Conversely, every such pair arises from a unique chain in R through $\mathfrak{p}$. But any maximal strictly ascending chain has length r, which implies the first equation. Also $\dim R/\mathfrak{m} = 0$, so $\dim R_{\mathfrak{p}} = \dim R$.

<u>Def.</u> R is <u>catenary</u> if given any two nested primes, all maximal chains of primes between the two have the same finite length.

<u>Thm.</u> Over a field, any f.gen. algebra is catenary.

<u>Pf.</u> Let $q \subset p \subset R$. Replacing R with R/q, can assume R is a domain, $q = 0$. Then any max chain has length $\dim R - \dim(R/p)$.