Lecture 13. Support

<u>Spectrum of a ring.</u> If R is a ring, its set of prime ideals is called the (prime) spectrum of R and denoted $\mathrm{Spec}(R)$. Let $\mathfrak{a}$ be an ideal. Let $V(\mathfrak{a})$ be the subset of primes containing $\mathfrak{a}$. We call $V(\mathfrak{a})$ the <u>variety</u> of $\mathfrak{a}$.

Let $\mathfrak{b}$ be another ideal. $\mathfrak{a} \subset \mathfrak{b} \Rightarrow V(\mathfrak{b}) \subset V(\mathfrak{a})$. Conversely, if $V(\mathfrak{b}) \subset V(\mathfrak{a})$ then $\mathfrak{a} \subset \sqrt{\mathfrak{b}}$. (Scheinnullstellensatz). $(\sqrt{\mathfrak{b}} = \bigcap_{\mathfrak{p} \supset \mathfrak{b}} \mathfrak{p})$.

Thus $V(\mathfrak{a}) = V(\mathfrak{b}) \Leftrightarrow \sqrt{\mathfrak{a}} = \sqrt{\mathfrak{b}}$. Also we get $V(\mathfrak{a}) \cup V(\mathfrak{b}) = V(\mathfrak{a} \cap \mathfrak{b}) = V(\mathfrak{a}\mathfrak{b})$.

(Exer. $\mathfrak{a} \subset \mathfrak{p}$ or $\mathfrak{b} \subset \mathfrak{p} \Leftrightarrow \mathfrak{a} \cap \mathfrak{b} \subset \mathfrak{p} \Leftrightarrow \mathfrak{a}\mathfrak{b} \subset \mathfrak{p}$

Also $\mathfrak{p} \supset \mathfrak{a}_\lambda \ \forall \lambda \Leftrightarrow \mathfrak{p} \supset \sum \mathfrak{a}_\lambda$. So

$$\bigcap V(\mathfrak{a}_\lambda) = V(\sum \mathfrak{a}_\lambda).$$

Also $V(R) = \emptyset$, $V(\langle 0 \rangle) = \mathrm{Spec}\, R$.

Thus $V(\mathfrak{a})$ can serve as closed sets of a topology on $\mathrm{Spec}\, R$. This topology is called Zariski topology.

Given $f \in R$, call the open set
$D(f) := \mathrm{Spec}\, R \setminus V(\langle f \rangle)$,
principal open set. These sets form a basis of the topology of $\mathrm{Spec}\, R$.

Indeed, $\forall \mathfrak{p} \not\supset \mathfrak{a}$ $\exists f \in \mathfrak{a} \setminus \mathfrak{p}$, so
$\mathfrak{p} \in D(f) \subset \operatorname{Spec} R - V(\mathfrak{a})$

Also, $f, g \notin \mathfrak{p} \Leftrightarrow fg \notin \mathfrak{p}$ $\forall f, g \in R$ and prime $\mathfrak{p}$, i.e. $D(f) \cap D(g) = D(fg)$.

A ring map $\varphi: R \to R'$ induces a set map $\operatorname{Spec}(\varphi): \operatorname{Spec}(R') \to \operatorname{Spec}(R)$ by
$(\operatorname{Spec} \varphi)(\mathfrak{p}') = \varphi^{-1}(\mathfrak{p}')$.

Notice $\varphi^{-1}(\mathfrak{p}') \supset \mathfrak{a} \Leftrightarrow \mathfrak{p}' \supset \mathfrak{a}R'$ so
$\operatorname{Spec}(\varphi)^{-1} V(\mathfrak{a}) = V(\mathfrak{a}R')$. Thus $\operatorname{Spec} \varphi$ is continuous, and Spec is a contravariant functor from ring to top. spaces.

E.g. The quot. map $R \to R/\mathfrak{a}$ gives rise to an embedding
$\operatorname{Spec}(R/\mathfrak{a}) \longrightarrow \operatorname{Spec} R$, whose image is
$\mathfrak{p} \longmapsto \mathfrak{p}$
$V(\mathfrak{a})$. Also, the localization map $R \to R_f$ gives rise to a topological embedding $\operatorname{Spec}(R_f) \longrightarrow \operatorname{Spec}(R)$ whose image is $D(f)$. (by the bijection between primes of R_f and primes of R not cont f).

Prop. R a ring, $X = \operatorname{Spec} R$. Then X is quasicompact: If $X = \bigcup_{\lambda \in \Lambda} U_\lambda$ with U_λ open, then $X = \bigcup_{i=1}^{n} U_{\lambda_i}$ for some $\lambda_i \in \Lambda$.

Pf. Say $U_\lambda = X \setminus V(\mathfrak{a}_\lambda)$. As $X = \bigcup_\lambda U_\lambda$, $\bigcap_\lambda V(\mathfrak{a}_\lambda) = \emptyset = V(\Sigma \mathfrak{a}_\lambda)$. So $\Sigma \mathfrak{a}_\lambda$ lies in no prime, and hence (i.e. $\Sigma \mathfrak{a}_\lambda = R$) So $\exists\ \lambda_1, \dots, \lambda_n$ and $f_i \in \mathfrak{a}_{\lambda_i}$ s.t. $\sum_i f_{\lambda_i} = 1$. Thus $R = \sum_{i=1}^{n} \mathfrak{a}_{\lambda_i}$, and $\emptyset = \bigcap V(\mathfrak{a}_{\lambda_i}) = V(\Sigma \mathfrak{a}_{\lambda_i})$, and $X = \bigcup U_{\lambda_i}$

Def. R a ring M an R-module. Its Support is the set $\operatorname{Supp} M = \{\mathfrak{p} \in \operatorname{Spec} R \mid M_\mathfrak{p} \neq 0\}$.

Prop. (1) $0 \to L \to M \to N \to 0$ exact $\Rightarrow \operatorname{Supp} M = \operatorname{Supp} L \cup \operatorname{Supp} N$

(2) let M_λ be submodules with $\Sigma M_\lambda = M$. Then $\bigcup \operatorname{Supp} M_\lambda = \operatorname{Supp} M$.

(3) $\operatorname{Supp} M \subset V(\operatorname{Ann} M)$, and they are equal if M is finitely generated.

Pf. (1) $\forall \mathfrak{p}$, $0 \to L_\mathfrak{p} \to M_\mathfrak{p} \to N_\mathfrak{p} \to 0$ is exact. $\Rightarrow M_\mathfrak{p} \neq 0$ iff $L_\mathfrak{p} \neq 0$, $N_\mathfrak{p} \neq 0 \Rightarrow (1)$.

(2) $M_\lambda \subset M$, so (1) gives $\bigcup_\lambda \operatorname{Supp} M_\lambda \subset \operatorname{Supp} M$.
Conversely, let $\wp \notin \bigcup_\lambda \operatorname{Supp} M_\lambda$. Then $\wp \notin \operatorname{Supp} M_\lambda \ \forall \lambda$, so $\forall \lambda \ (M_\lambda)_\wp = 0 \Rightarrow$ $(\oplus M_\lambda)_\wp = \oplus (M_\lambda)_\wp = 0$. But $\bigoplus_\lambda M_\lambda \twoheadrightarrow M$, so $M_\wp = 0$ and $\wp \in \operatorname{Supp} M$.

(3) Let $\wp \in \operatorname{Supp} M \Rightarrow M_\wp \neq 0$. So $\exists m \in M$ $\forall f \in R - \wp$ $sm \neq 0$. Hence $\operatorname{Ann}(M) \subset \wp \Rightarrow$ $\wp \in V(\operatorname{Ann} M)$. Conversely, let $\wp \in V(\operatorname{Ann} M)$, i.e. $\wp \supset \operatorname{Ann}(M)$. Let $m_1, \dots, m_n$ generators of M. Supp. $M_\wp = 0$. Then $\exists s_i \notin \wp$, $i = 1, \dots, n$ $s_i m_i = 0 \Rightarrow \prod s_i \cdot m_j = 0 \ \forall j$ so $f = \prod s_j$ satisfies $s \in \operatorname{Ann}(M) \Rightarrow \Leftarrow$ (as $s \notin \wp$).
Thus $M_\wp \neq 0$ and $\wp \in$ ~~Ann~~ $\operatorname{Supp} M$.

Def. R a ring, $x \in R$. Say x nilpotent on M if $\exists n$ s.t. $x^n M = 0$ $\forall m \in M$; i.e., $x \in \sqrt{\operatorname{Ann} M}$. Denote the set of nilpotents by $\operatorname{nil}(M)$, i.e. $\operatorname{nil}(M) = \sqrt{\operatorname{Ann} M}$.

Prop. R ring, M f.g. module. Then
$$\operatorname{nil}(M) = \bigcap_{\wp \in \operatorname{Supp} M} \wp$$

Pf. First $\operatorname{nil} M = \bigcap_{\wp \supset \operatorname{Ann} M} \wp$ by Schein nullstellensatz.
But $\wp \supset \operatorname{Ann} M \Leftrightarrow \wp \subset \operatorname{Supp} M$.

<u>Prop.</u> R a ring, M, N modules. Then

$\mathrm{Supp}(M \otimes_R N) \subset \mathrm{Supp} M \cap \mathrm{Supp} N$,

with equality if M, N fin. gen.

<u>Pf.</u> $(M \otimes_R N)_\mathfrak{p} = M_\mathfrak{p} \otimes_{R_\mathfrak{p}} N_\mathfrak{p}$, so the inclusion holds. The opposite incl. follows from lemma

<u>L.</u> A local ring with res. field k, M, N f.g. mod. Then $M \otimes_A N \neq 0 \Leftrightarrow M \neq 0, N \neq 0$.

<u>Pf.</u> Nakayama: $M \otimes_A N \neq 0 \Leftrightarrow M \otimes_A N \otimes_A k \neq 0$

But this is $M \otimes_A k \otimes_k N \otimes_A k$. And this is $\neq 0$ $\Leftrightarrow M \otimes_A k \neq 0, N \otimes_A k \neq 0$, which is equiv. $M \neq 0, N \neq 0$ for f.g. M, N.

<u>Prop.</u> R a ring, M a module. TFAE:

(1) $M = 0$

(2) $\mathrm{supp} M = \emptyset$

(3) $M_m = 0$ for each max. ideal $m \subset R$.

<u>Pf.</u> (1) $\Rightarrow S^{-1}M = 0 \ \forall S \Rightarrow$ (2)

Also if $M \neq 0$, then can take $m \in M$, $m \neq 0$ take a max id $m \supset \mathrm{Ann}\, m$ and $M_m \neq 0$, so (3) $\Rightarrow$ (1).

($\forall f \notin m$ s.t. $fm \neq 0$ so $M_m \neq 0$).

Also (2) $\Rightarrow$ (3) since any ~~prime is~~ max. ideal is prime.

<u>Prop.</u> A sequence $L \xrightarrow{\alpha} M \xrightarrow{\beta} N$ is exact $\Leftrightarrow$ localization $L_m \to M_m \to N_m$ is exact for each max. ideal m.

Pf.

"If" follows from the fact that localization is exact. Converse: ~~Clearly~~ We have $\mathrm{Im}(\beta_m \alpha_m) = 0$, so $(\mathrm{Im}(\beta\alpha))_m = 0$

So $\mathrm{Im}(\beta\alpha) = 0$ by the above, so $\beta\alpha = 0$, i.e. $\mathrm{Im}\,\alpha \subset \mathrm{Ker}\,\beta$. Let $H = \mathrm{Ker}\,\beta / \mathrm{Im}\,\alpha$.

Then $H_m = \mathrm{Ker}\,\beta_m / \mathrm{Im}\,\alpha_m$ by exactness of localization. So $H_m = 0$ Hence $H = 0$ by the above.

Prop. A a semilocal ring, $m_1, .., m_n$ max. ideals, M, N fin. presented modules. Supp. $M_{m_i} \simeq N_{m_i}$ $\forall i$, then $M \cong N$.

Pf. $\forall i$ let $\psi_i : M_{m_i} \cong N_{m_i}$. Then $\exists s_i \in A \setminus m_i$ and $\varphi_i : M \to N$ s.t. $(\varphi_i)_{m_i} = s_i \psi_i$ $\left(\frac{\varphi_i}{s_i} = \psi_i\right)$ (localization gives isom. at the level of morphisms). But $\bigcap_{j \neq i} m_j \subset m_i$,

so $\exists x_i \in \bigcap_{j \neq i} m_j$ with $x_i \notin m_i$. Let

$\gamma = \sum x_i \varphi_i$.

$\forall j$ define $\alpha_j = x_j s_j \psi_j$. Then $\alpha_{m_j} : M_{m_j} \cong N_{m_j}$ as x_j and s_j are units of R_{m_j}.

Let $\beta_j = \sum_{j \neq i} \alpha_j$. Then $\beta_j(M_{m_j}) \subset m_j N_{m_j}$, as $x_i \in m_j$ for $j \neq i$.

Now $\gamma = \alpha_j + \beta_j$, so γ_{m_j} is an isom.

($\alpha_j + \beta_j$ is a "small perturbation" of α_j).

Hence γ is an isom.

<u>Prop.</u> R a ring, M an R-module. Then M is flat$/R$ iff $\forall$ maximal $m \subset R$, M_m is flat over R_m.

<u>Pf.</u> M flat$/R \Rightarrow M_m = M \otimes_R R_m$ is flat over R_m. (by cancellation). Conversely, assume M_m is flat over R_m $\forall m$. Let $\alpha : N \to N'$ be an injection of R-modules. Then α_m is inj since $\otimes_R R_m$ is exact. So $M_m \otimes_{R_m} \alpha_m$ is injective. So $(M \otimes \alpha)_m$ is injective. Thus by the above $M \otimes \alpha$ is injective, so M is flat. ∎

<u>Def.</u> R a ring, M a module. Say that M is <u>locally finitely generated</u> if $\forall \mathfrak{p} \in \operatorname{Spec} R$ has a nbhd on which M becomes fin. gen., i.e. $\exists f \in R - \mathfrak{p}$ s.t. M_f is f.g. over R_f. It suffices that f exist for each max. ideal m since each $\mathfrak{p}$ lies in some m. Similarly, we can define the property of being locally finitely presented, locally free of finite rank, locally free of rank n

<u>Prop.</u> Let M be a R-module.

(1) If M is locally f.g $\Rightarrow$ it is f.g.

(2) If M is locally f.pres $\Rightarrow$ it is f.pres.

<u>Pf.</u> $\exists f_1, \dots, f_n \in R$ s.t. $\bigcup D(f_i) = \operatorname{Spec} R$

and $m_{ij} \in M$ (fin. many) s.t. $\frac{m_{ij}}{f_i^{n_{ij}}}$ generate M_{f_i}

Then $\frac{m_{ij}}{1}$ also gen. M_{f_i} $\forall i$

Given any max. ideal m,

$\exists i$ s.t. $f_i \notin m$. Let S_i be the image

of $R \setminus m$ in R_{f_i}. Then $M_m = S_i^{-1} M_{f_i}$.

So $\frac{m_{ij}}{1}$ generate M_m.

<u>Lemma.</u> Elements $m_\lambda \in M$ generate M iff they generate after localiz. by all max ideals.

<u>Pf.</u> Clearly, generate $\Rightarrow$ generate after localiz. Also if m_λ generate after localiz. then $R_m^\wedge \to M_m \to 0$ is exact $\forall m$, so $R^\wedge \to M$ is exact.

By lemma, this means that m_{ij} generate M, which gives (1).

To prove (2), assume M loc. fin. pres. Then M is f. gen. by (1). So have a surjection $R^k \to M$. Let K be kernel. Then K is loc. f.g. So K is f.g., as desired. (get (2)).

<u>Thm</u>. The foll cond on R are equiv:

(1) P f.g. proj.

(2) P f. pres. and flat

(3) P f. pres, P_m free over R_m at each max. ideal m.

(4) P loc. free of fin. rank.

(5) P f. gen, $\forall \mathfrak{p} \in \operatorname{Spec} R$ $\exists f, n$ s.t. $\mathfrak{p} \in D(f)$, and P_g is free of rank n for each $g \in D(f)$.

<u>Pf.</u> (1) $\Rightarrow$ (2) proved before.

(2) $\Rightarrow$ (3) Let m max. ideal. R_m local, P_m fin. pres., and flat, so by a result from before, P_m is free.

(3) $\Rightarrow$ (1) Fix a surjection $\alpha : M \to N$. Then $\alpha_m : M_m \to N_m$ is surjective. So $Hom(P_m, \alpha_m)$: $Hom(P_m, M_m) \to Hom(P_m, N_m)$ is surjective. But $Hom(P_m, \alpha_m) = Hom(P, \alpha)_m$, as P is

finitely presented. Thus $\mathrm{Hom}(P,\alpha)_m$ is surjective for all m, so $\mathrm{Hom}(P,\alpha)$ is surjective. So P is proj. ($\mathrm{Hom}(P,?)$ exact). This implies (1).

~~Also~~ (3) $\Rightarrow$ (4) Given a prime $\mathfrak{p}$, let $m \supset \mathfrak{p}$ be a max. ideal. ~~Since~~ P_m is free, of finite rank since f. gen. Then $\exists f \in R \setminus m$ s.t. P_f is free of f. rank over $R_f \Rightarrow$ (4).

(4) $\Rightarrow$ (5), (3) P loc. fin. pres. So P is f. pres. Also $\forall\, \mathfrak{p} \in \mathrm{Spec}(R)$, $\exists f \in R \setminus \mathfrak{p}$ s.t. P_f is free of fin. rk over R_f. Given $g \in D(f)$, let S be the image of $R \setminus g$ in R_f. Then $P_g = S^{-1}P_f$. So P_g is free of rank n over $R_g \Rightarrow$ (5). Also taking $p = q = m$, get (3).

(5) $\Rightarrow$ (4). Given $\mathfrak{p} \in \mathrm{Spec}\, R$, let f, n be given by (5). Let $\frac{p_1}{f^{k_1}}, \dots, \frac{p_n}{f^{k_n}}$ be the free basis of $P_{\mathfrak{p}}$ over $R_{\mathfrak{p}}$. Then p_i define a map $R^n \xrightarrow{\alpha} P$, and α_p is bijective, in partic. surjective. As P is f. gen., $\exists g$ s.t. $R^n_g \to P_g$ is surjective (as we showed before).

So $\alpha_q : R_q^n \to P_q$ is surj. $\forall q \in D(g)$.
If also $q \in D(f)$, then $R_q^n \cong P_q$ so α_q is bij. (any surj. map of free modules is bijective).
Let $h = fg$ Then $D(f) \cap D(g) = D(h)$.
Now $D(h) = \mathrm{Spec}(R_h)$. So $\alpha_h : R_h^n \to P_h$ is bijective $\Rightarrow (4)$.