Lecture 12. Localization of modules

<u>Prop.</u> $R \supset S$ mult. subset, M an R-module. Then M has a compatible $S^{-1}R$-module structure $\Leftrightarrow$ $\forall s \in S$, $\mu_s : M \to M$ (mult. by s) is bijective. If so, then the $S^{-1}R$-module str. on M is unique

<u>Pf.</u> easy (in the notes).

<u>Localization of modules</u>. $R \supset S$ mult. subset, M an R-module. Define a rel. on $M \times S$ by $(m,s) \sim (n,t)$ if $\exists u \in S$ s.t. $utm = usn$. This is an equivalence.

Let $S^{-1}M$ be the set of equiv. classes, denote by $\frac{m}{s}$ the class of (m,s).

Then $S^{-1}M$ is an $S^{-1}R$-module. We call $S^{-1}M$ the <u>localization</u> of M at S.

E.g. if $\mathfrak{a} \subset R$ is an ideal, then $S^{-1}\mathfrak{a} = \mathfrak{a}S^{-1}R$ as shown above.

Also $S^{-1}\mathfrak{a}M = \mathfrak{a}S^{-1}M$. Also if $R \to R'$ then $S^{-1}R'$ as an $S^{-1}R$-module is what we defined before.

Let $\varphi_S : M \to S^{-1}M$, $\varphi_S(m) = \frac{m}{1}$.

Note that $\mu_s : S^{-1}M \circlearrowleft$ is bij $\forall s \in S$ by the above.

If $S = \{f^n / n \geq 0\}$ for $f \in R$ then $S^{-1}M$ is the <u>localization</u> of M at f, and we denote $S^{-1}M$ by M_f (with $\varphi_f = \varphi_S$).

Similarly, if $S = R - \mathfrak{p}$ for a prime $\mathfrak{p}$ then call $S^{-1}M$ the <u>localization</u> of M at $\mathfrak{p}$ and set $M_{\mathfrak{p}} = S^{-1}M$, $\varphi_{\mathfrak{p}} = \varphi_S$.

<u>Thm.</u> (UMP) $S^{-1}M$ is universal among $S^{-1}R$-modules equipped with an R-linear map from M.

<u>Pf.</u> Easy (in the notes).

~~Given M, define a functor $S \to R$-mod~~

<u>Functoriality.</u> $R \supset S$, $\alpha : M \to N$ an R-linear map. Then $\varphi_S \alpha$ carries M to the $S^{-1}R$-mod $S^{-1}N$. So UMP yields a unique $S^{-1}R$-linear map $S^{-1}\alpha$ s.t.

$$\begin{array}{ccc} M & \xrightarrow{\alpha} & N \\ \downarrow \varphi_S & \circlearrowleft & \downarrow \varphi_S \\ S^{-1}M & \xrightarrow{S^{-1}\alpha} & S^{-1}N \end{array}.$$

Thus we get $\mathrm{Hom}_R(M, N) \to \mathrm{Hom}_{S^{-1}R}(S^{-1}M, S^{-1}N)$.

So $S^{-1}\cdot$ is a linear functor from R-mod to $S^{-1}R$-mod.

<u>Thm.</u> $R \supset S$. Then the functor $S^{-1}\cdot$ is left adjoint to the functor of restr. of scalars.

<u>Pf.</u> $\mathrm{Hom}_{S^{-1}R}(S^{-1}M, N) = \mathrm{Hom}_R(M, N)$ by univ. property.

<u>Cor.</u> $R \supset S$. Then $S^{-1}\cdot$ preserves direct limits (or equivalently, $\oplus$ and cokernels)

<u>Pf.</u> True because it is left adjoint.

<u>Cor.</u> $S^{-1}M \cong S^{-1}R \otimes_R M$ as functors.

<u>Pf.</u> As this functor preserves direct sums and cokernels, (=right exact) this follows from Watt's thm. Alternatively, both functors are left adj. to restr. of scalars.

<u>Def.</u> $R \supset S$, M an R-module. If $N \subset M$, saturation N^S is $\{m \in M \mid \exists s \in S \ sm \in N\}$. If $N = N^S$ then N saturated.

<u>Prop.</u> $R \supset S$, M R-mod, $N, P \subset M$ submodules K an $S^{-1}R$-submod of $S^{-1}M$.

(1) $N^S \subset M^S$ submodule, $S^{-1}N \subset S^{-1}M$.

(2) $\varphi_S^{-1}K = (\varphi_S^{-1}K)^S$ and $K = S^{-1}(\varphi_S^{-1}K)$

(3) $\varphi_S^{-1}(S^{-1}N) = N^S$, and $\ker \varphi_S = 0^S$. $(\varphi_S : N \to S^{-1}N)$

(4) $(N^S)^S = N^S$, $S^{-1}(S^{-1}N) = S^{-1}N$

(5) $N \subset P \Rightarrow N^S \subset P^S$, $S^{-1}N \subset S^{-1}P$.

(6) $(N \cap P)^S = N^S \cap P^S$, $S^{-1}(N \cap P) = S^{-1}N \cap S^{-1}P$.

(7) $(N+P)^S \supset N^S + P^S$, $S^{-1}(N+P) = S^{-1}N + S^{-1}N$

<u>Pf.</u> Straightforward applic of def.

<u>Thm.</u> Exactness of localiz.

$R \supset S$. Then $S^{-1}\cdot$ is an exact functor

Pf. S^{-1} preserves injections and cokernels

Cor. $S^{-1}R$ is a flat R-module.

Pf. This holds since $S^{-1}M = S^{-1}R \otimes_R M$.

Cor. $R \supset S$, $\mathfrak{a}$ ideal, M R-mod.
Then $S^{-1}(M/\mathfrak{a}M) = S^{-1}M/S^{-1}\mathfrak{a}M = S^{-1}M/\mathfrak{a}S^{-1}M$.

Cor. $R \supset \mathfrak{p}$ prime. Then $\mathrm{Frac}(R/\mathfrak{p}) = R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$.

Pf. $\mathrm{Frac}(R/\mathfrak{p}) = (R/\mathfrak{p})_\mathfrak{p} = R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$.

Prop. $R \supset S$, M an R-module.

(1) Let $m_1, \dots, m_n \in M$. If M is fin. gen and if $\frac{m_i}{1} \in S^{-1}M$ generate it over $S^{-1}R$, then $\exists f \in S$ s.t. $\frac{m_i}{1} \in M_f$ generate over R_f.

(2) Assume M is finitely presented, and $S^{-1}M$ is a free $S^{-1}R$-module of rank n. Then $\exists h \in S$ s.t. M_h is a free R_h-module of rank n.

Pf. For (1), let $\alpha: R^n \to M$, $\alpha(e_i) = m_i$. Let $C = \mathrm{Coker}(\alpha)$. Then $S^{-1}C = \mathrm{Coker}(S^{-1}\alpha)$ (as S^{-1} preserves cokernels). Ass. $\frac{m_i}{1}$ generate $S^{-1}M$ over $S^{-1}R$. Then $S^{-1}\alpha$ is surjective. So $S^{-1}C = 0$.

Now assume M is fin. generated. Then so is C.

We have generators $c_1, \dots, c_r$ of C. $\forall i\ \exists s_i \in S$ s.t. $s_i c_i = 0$. Then take $f = \prod s_i$, and get $f c_i = 0 \Rightarrow C_f = 0$.

So α_f is surjective. So $\frac{m_i}{1}$ generate M_f over R_f. Thus get (1).

(2): For (2), let $\frac{m_1}{s_1}, \dots, \frac{m_n}{s_n}$ be a free basis of $S^{-1}M$ over $S^{-1}R$. Then so is $\frac{m_1}{1}, \dots, \frac{m_n}{1}$, as $\frac{1}{s_i}$ are units.

For α, C as above, let $K = \operatorname{Ker}\alpha$.

Then $S^{-1}K = \operatorname{Ker}(S^{-1}\alpha)$, as S^{-1} commutes with kernels. But $S^{-1}\alpha$ is bijective. So $S^{-1}K = 0$ and $S^{-1}C = 0$.

Since M is f.gen, so is C. So, as above, $\exists f \in S$ s.t. $C_f = 0$. Then $0 \to K_f \to R_f^n \to M_f \to 0$ is a SES. Take a finite presentation $R^p \to R^q \to M \to 0$. This yields a fin pres. $R_f^p \to R_f^q \to M_f \to 0$. So K_f is a f.gen. R_f-module. Let $S_1 \subset R_f$ be the image of S. Then $S_1^{-1}(K_f) = S^{-1}K$. But $S^{-1}K = 0$. So $\exists \frac{g}{1} \in S_1$ s.t. $(K_f)_{\frac{g}{1}} = 0$.

Let $h = fg$. Then $K_h = 0$. But $C_f = 0$ implies $C_h = 0$. So $\alpha_h : R_h^n \to M_h$ is an isomorphism.

<u>Prop.</u> $R \supset S$, M and N modules.

Then there is a canonical homom.

$$\sigma : S^{-1}\operatorname{Hom}_R(M, N) \to \operatorname{Hom}_{S^{-1}R}(S^{-1}M, S^{-1}N)$$

Further, σ is inj. if M is fin. gen. and an isom if M is finitely presented.

<u>Ex</u>. $R=\mathbb{Z}$, $S=\mathbb{Z}\setminus 0$. $M=\mathbb{Q}/\mathbb{Z}$. M is faithful so $\mu_R : R \to End_R(M,M)$ is injective.

But $S^{-1}R=\mathbb{Q}$. So $S^{-1}Hom_R(M,M)\neq 0$.

But $S^{-1}M=0$ so $\sigma(M,M)$ is not inj.

(note that M is not f.gen).

<u>Ex</u>. $R=\mathbb{Z}$, $S=\mathbb{Z}\setminus 0$, $M_n=\mathbb{Z}/\langle n\rangle$, $n\geq 2$.

Then $S^{-1}M_n=0$ $\forall n$. But $(1,\dots 1)/_1 \neq 0$ in $S^{-1}(\prod_n M_n)$, as the k-th comp. of $m(1,\dots,1)$ is nonzero. Also $S^{-1}\mathbb{Z}=\mathbb{Q}$. So get

$\mathbb{Q}\otimes_{\mathbb{Z}}(\prod M_n)\neq\prod(\mathbb{Q}\otimes_{\mathbb{Z}}M_n)$, while $\mathbb{Q}\otimes_{\mathbb{Z}}(\oplus M_n)=$

$=\oplus\mathbb{Q}\otimes_{\mathbb{Z}}M_n$.