Lecture 11. Localization

R ring $S \subset R$ mult. subset.

Relation on $R \times S$: $(x,s) \sim (y,t)$ if $\exists u \in S$ s.t. $xtu = ysu$. ("$\frac{x}{s} = \frac{y}{t}$") (need a because R may not be a domain)

This relation is an equiv. rel.

Reflexive: $u=1$, obv. symmetric

$(x,s) \sim (y,t) \sim (z,r)$

$yrv = ztv$, $v \in S$ $\Rightarrow$

$\underset{\vee}{x}tu\underset{\vee}{z}v = ysuzv = \underset{\vee\vee}{z}\underset{\vee}{s}utv$ so $(x,s) \sim (z,t)$

> Ex. $R = \mathbb{C}[x,y]/(xy)$. $S = 1, x, x^2, \dots$
> $S^{-1}R = \mathbb{C}[x, x^{-1}]$.
> (as $y=0$ in $S^{-1}R$)

Denote by $S^{-1}R$ the set of equiv. classes, and by $\frac{x}{s}$ the class of (x,s).

Let $\frac{x}{s} \cdot \frac{y}{t} = \frac{xy}{st}$. It's easy to check it is well defined.

Let $\frac{x}{s} + \frac{y}{t} = \frac{xt+ys}{st}$. This is also well defined.

So we get a ring $S^{-1}R$, <u>ring of fractions with respect to S</u> or <u>localization at S</u>

Let $\varphi_S : R \to S^{-1}R$ be the map given by $\varphi_S(x) = \frac{x}{1}$. Then φ_S is a ring map, and it carries elements of S to units.

<u>Total quotient ring</u> $R \supset S_0$ - set of nonzero divisors. Then S_0 is a saturated mult. subset.

The map $\varphi_{S_0} : R \to S_0^{-1}R$ is injective as if $\varphi_{S_0}(x) = 0$ then $sx = 0$ for some $s \in S \Rightarrow x = 0$.

<u>Def</u>. $S_0^{-1}R$ is called the <u>total quotient ring</u> of R.

If $S \subset S_0$ then $R \subset S^{-1}R \subset S_0^{-1}R$.

Suppose R is a domain. Then $S_0 = R \setminus 0$ so the total quotient ring is just the fraction field $\operatorname{Frac} R$, and $S^{-1}R$ is a domain for any $S \subset S_0$.

But if $S = R$ (or even $S \ni 0$) then can take $u = 0$, so all pairs are equivalent and $S^{-1}R = 0$.

<u>Thm</u>. (UMP of $S^{-1}R$). $S^{-1}R$ is universal among algebras which render all $s \in S$ units. I.e. $\operatorname{Hom}(S^{-1}R, R') = \left\{ \begin{array}{l} f \in \operatorname{Hom}(R, R') \\ f(S) \subset \operatorname{Units}(R') \end{array} \right\}$.

This is true even if R' is noncommutative.

<u>Pf</u>. In the notes.

<u>Cor</u>. R a ring, $S \subset R$ mult. subset. Then the canonical map $\varphi_S : R \to S^{-1}R$ is an isom. $\Leftrightarrow$ S consists of units.

Pf. If φ_S is an isomorphism then S consists of units, since so does $\varphi_S(S)$. Conversely, if S consists of units then the identity map $R \to R$ has UMP that characterizes φ_S, so φ_S is an isom.

Def. R a ring, $f \in R$, $S = \{f^n, n \geq 0\}$. We call $S^{-1}R$ the localization of R at f, and set $R_f = S^{-1}R$, $\varphi_f = \varphi_S$. Also write $R_f = R[\frac{1}{f}]$ ~~If R is a domain~~

Prop. $R_f \cong R[X]/\langle 1 - fX \rangle$

Pf. $R' \overset{\text{def}}{=} R[X]/\langle 1 - fX \rangle$, $\varphi: R \to R'$ the canonical map. Let's show R' has the UMP characterizing localization. Let $x \in R'$ be the residue of X. Then $1 - x\varphi(f) = 0$. So $\varphi(f)$ is a unit. So $\varphi(f^n)$ is a unit for $n \geq 0$. Now, let $\psi: R \to R''$ is a homomorphism carrying f to a unit. Define $\theta: R[X] \to R''$ by $\theta|_R \to \psi$ and $\theta(X) = \psi(f)^{-1}$. Then $\theta(1 - fX) = 0$, so θ factors via a homom. $\rho: R' \to R''$ s.t. $\psi = \rho\varphi$. Also ρ is unique since every element of R' is a polynomial of x and $\rho(x) = \psi(f)^{-1}$.

<u>Prop</u>: $R \supset S$ mult. subset in a ring R, $\mathfrak{a} \subset R$ an ideal.

(1) $\mathfrak{a}S^{-1}R = \{ a/s \in S^{-1}R \mid a \in \mathfrak{a}, s \in S \}$.

(2) $\mathfrak{a} \cap S \neq 0 \Leftrightarrow \mathfrak{a}S^{-1}R = S^{-1}R \Leftrightarrow \varphi_S^{-1}(\mathfrak{a}S^{-1}R) = R$.

<u>Pf.</u> (1) Let $a, b \in \mathfrak{a}$, $\frac{x}{s}, \frac{y}{t} \in S^{-1}R$. Then $\frac{ax}{s} + \frac{by}{t} = \frac{axt+bys}{st} \in RHS$, so $\mathfrak{a}S^{-1}R \subset \{ \frac{a}{s}, a \in \mathfrak{a}, s \in S \}$, and get (1)

(2) If $s \in \mathfrak{a} \cap S$ then $\mathfrak{a}S^{-1}R \ni \frac{s}{s} = 1$, so $\mathfrak{a}S^{-1}R = S^{-1}R$; This implies, in turn that $\varphi_S^{-1}(\mathfrak{a}S^{-1}R) = R$. Now, if $\varphi_S^{-1}(\mathfrak{a}S^{-1}R) = R$. Then $\mathfrak{a}S^{-1}R \ni 1$. So (1) yields $a \in \mathfrak{a}$ and $s \in S$ s.t. $1 = \frac{a}{s}$. Thus $\exists\, t \in S$ s.t. $at = st$. but $\mathfrak{a} \ni at$, $st \in S$. So $\mathfrak{a} \cap S \neq \emptyset$, and we get (2).

<u>Def.</u> R a ring, S mult. subset, $\mathfrak{a} \subset R$. The <u>saturation</u> of $\mathfrak{a}$ with respect to S, $\mathfrak{a}^S$, is

$$\mathfrak{a}^S = \{ a \in R \mid \exists s \in S \text{ with } as \in \mathfrak{a} \}.$$

If $\mathfrak{a} = \mathfrak{a}^S$, say $\mathfrak{a}$ is <u>saturated</u>.

<u>Ex</u>: $\mathfrak{a} = \langle x \rangle \subset \mathbb{C}[x]$
$S = \langle 1, x, x^2, \dots \rangle$
$\mathfrak{a}^S = \langle 1 \rangle = \mathbb{C}[x]$

<u>Prop.</u> $R \supset S$ mult. subset, $\mathfrak{a}$ an ideal. Then:

(1). $\operatorname{Ker} \varphi_S = \langle 0 \rangle^S$

(2). $\mathfrak{a} \subset \mathfrak{a}^S$

(3). $\mathfrak{a}^S \subset R$ is an ideal.

<u>Pf.</u> $\frac{a}{1} = 0$ if and only if $\exists s \in S$ s.t. $as = 0 \Rightarrow (1)$

$1 \in S \Rightarrow (2)$. If $as, bt \in \mathfrak{a}$ then $ast, bst \in \mathfrak{a} \Rightarrow (a+b)st \in \mathfrak{a}$ and if $x \in R$ then $xas \in \mathfrak{a} \Rightarrow (3)$

<u>Prop.</u> $R \supset S$ mult. subset.

(1) Let $\mathfrak{b} \subset S^{-1}R$ be an ideal. Then

(a) $\varphi_S^{-1}\mathfrak{b} = (\varphi_S^{-1}\mathfrak{b})^S$ and (b) $\mathfrak{b} = (\varphi_S^{-1}\mathfrak{b}) \cdot S^{-1}R$.

(i.e. $\varphi_S^{-1}\mathfrak{b}$ is saturated)

(2) $\mathfrak{a} \subset R$ ideal. Then $\varphi_S^{-1}(\mathfrak{a}S^{-1}R) = \mathfrak{a}^S$

(3) Let $\mathfrak{p} \subset R$ be a prime with $\mathfrak{p} \cap S = \emptyset$. Then

(a) $\mathfrak{p} = \mathfrak{p}^S$ (i.e. $\mathfrak{p}$ saturated) and (b) $\mathfrak{p}S^{-1}R$ is prime.

Pf. (1)(a): Take $a \in \mathfrak{a}$ and $s \in S$ with $as \in \varphi_S^{-1}\mathfrak{b}$.
Then $\frac{as}{1} \in \mathfrak{b}$. So $\frac{a}{1} \in \mathfrak{b}$ as $1/s \in S^{-1}R$. Thus $a \in \varphi_S^{-1}\mathfrak{b}$.
So $(\varphi_S^{-1}\mathfrak{b})^S \subseteq \varphi_S^{-1}\mathfrak{b}$. The opposite incl. is obvious, so get 1(a).
(1)(b) Take $\frac{a}{s} \in \mathfrak{b}$. Then $\frac{a}{1} \in \mathfrak{b}$, so $a \in \varphi_S^{-1}\mathfrak{b}$. Thus $\frac{a}{1} \cdot \frac{1}{s} \in (\varphi_S^{-1}\mathfrak{b}) S^{-1}R$.
So $\mathfrak{b} \subset \varphi_S^{-1}\mathfrak{b} \cdot S^{-1}R$. Now let $a \in \varphi_S^{-1}\mathfrak{b}$. Then $\frac{a}{1} \in \mathfrak{b}$.
So $\mathfrak{b} \supset (\varphi_S^{-1}\mathfrak{b})(S^{-1}R)$. So (1)(b) holds also.

(2): Take $a \in \mathfrak{a}^S$. $\exists s \in S$ s.t. $as \in \mathfrak{a}$. But $\frac{a}{1} = \frac{as}{1} \cdot \frac{1}{s}$.
So $\frac{a}{1} \in \mathfrak{a}S^{-1}R$. Thus $\varphi_S^{-1}(\mathfrak{a}S^{-1}R) \supseteq \mathfrak{a}^S$. Now take $x \in \varphi_S^{-1}(\mathfrak{a}S^{-1}R)$.
Then $\frac{x}{1} = \frac{a}{s}$ with $a \in \mathfrak{a}$ and $s \in S$. by a result above.
So $\exists t \in S$ s.t. $xst = at \in \mathfrak{a}$. So $x \in \mathfrak{a}^S$, and $\varphi_S^{-1}(\mathfrak{a}S^{-1}R) \subset \mathfrak{a}^S \Rightarrow (2)$.

(3): $\mathfrak{p} \subset \mathfrak{p}^S$ as $1 \in S$. Conversely, if $sa \in \mathfrak{p}$ with $s \in S \subset R \setminus \mathfrak{p}$
then $a \in \mathfrak{p}$ as $\mathfrak{p}$ is prime. So (a) holds.
As for (b), note first that $\mathfrak{p}S^{-1}R \neq S^{-1}R$ as $\varphi_S^{-1}(\mathfrak{p}S^{-1}R) = \mathfrak{p}^S = \mathfrak{p}$
by (2) and (3a) and $1 \notin \mathfrak{p}$ (as $\mathfrak{p} \cap S = \emptyset$). Second, say $\frac{a}{s} \cdot \frac{b}{t} \in \mathfrak{p}S^{-1}R$.
Then $ab \in \varphi_S^{-1}(\mathfrak{p}S^{-1}R)$, and the latter is $\mathfrak{p}^S$ by (2), hence $\mathfrak{p}$
by (3a). Hence $ab \in \mathfrak{p}$ so $a \in \mathfrak{p}$ or $b \in \mathfrak{p}$ as $\mathfrak{p}$ is prime.
So $\frac{a}{s}$ or $\frac{b}{t} \in \mathfrak{p}S^{-1}R$. Thus $\mathfrak{p}S^{-1}R$ is prime $\Rightarrow (3)$.

Cor. $R \supset S$ mult. subset.

(1) $\mathfrak{a} \mapsto \mathfrak{a}S^{-1}R$ is an inclusion-preserving bijection
$\{\text{ideals } \mathfrak{a} \subset R \text{ with } \mathfrak{a} = \mathfrak{a}^S\} \xrightarrow{\sim} \{\text{ideals in } S^{-1}R\}$.
The inverse is $\mathfrak{b} \to \varphi_S^{-1}\mathfrak{b}$.

(2) $\mathfrak{p} \mapsto \mathfrak{p}S^{-1}R$ is an inclusion-preserving bij:
$\{\text{primes in } R \text{ with } \mathfrak{p} \cap S = \emptyset\} \xrightarrow{\sim} \{\text{primes in } S^{-1}R\}$.
The inverse is $\mathfrak{q} \to \varphi_S^{-1}\mathfrak{q}$.

Pf. (1) is clear from the above. For (2), we also need to use that preimage of a prime ideal is prime.

Def. $R \supset \mathfrak{p}$ prime. Let $S = R \setminus \mathfrak{p}$. We call $S^{-1}R$ the localization of R at $\mathfrak{p}$, and set $R_\mathfrak{p} = S^{-1}R$ and $\varphi_\mathfrak{p} = \varphi_S$.

Prop. $R \supset \mathfrak{p}$ prime. Then $R_\mathfrak{p}$ is local with max ideal $\mathfrak{p}R_\mathfrak{p}$.

Pf. Let $\mathfrak{b} \subset R_\mathfrak{p}$ be a proper ideal. Then $\varphi_\mathfrak{p}^{-1}\mathfrak{b} \subset \mathfrak{p}$ by the above (otherwise $\varphi_\mathfrak{p}^{-1}\mathfrak{b} \cap S \neq \emptyset$, so $\varphi_\mathfrak{p}^{-1}\mathfrak{b} \cdot S^{-1}R = S^{-1}R$, but $\varphi_\mathfrak{p}^{-1}\mathfrak{b} \cdot S^{-1}R = \mathfrak{b}$). So $\mathfrak{b} \subset \mathfrak{p}R_\mathfrak{p}$ since ~~$\varphi_\mathfrak{p}^{-1}(\mathfrak{p}R_\mathfrak{p}) \subset \mathfrak{p}$ (in the same way as above) and~~ $\varphi_\mathfrak{p}^{-1}(\mathfrak{b}) \subset \mathfrak{p} \subset \varphi_\mathfrak{p}^{-1}(\mathfrak{p}R_\mathfrak{p})$ (as we have an inclusion-preserving bijection). So $\mathfrak{p}R_\mathfrak{p} \subset R_\mathfrak{p}$ is the only max. ideal.

~~Conversely, let $x \notin \mathfrak{p}$. Then $s/x \in R_\mathfrak{p}$. So $\frac{s}{x}$ is a unit in $R_\mathfrak{p}$ if and only if $x \notin \mathfrak{p}$, i.e., iff $\frac{x}{s} \notin \mathfrak{p}R_\mathfrak{p}$. So the nonunits of $R_\mathfrak{p}$ form $\mathfrak{p}R_\mathfrak{p}$, which is an ideal, giving the desired statement.~~

$R \supset S$ mult. subset, R' an R-algebra. Then we can generalize the definition of localization as follows. Define a rel. on $R' \times S$ by $(x,s) \sim (y,t)$ if $\exists\, u \in S$ with $xtu = ysu$. This is an equiv. rel. Let $S^{-1}R'$ be the set of equiv. classes and by $\frac{x}{s}$ the class of (x,s). This is an $S^{-1}R$-algebra.

We call $S^{-1}R'$ the <u>localization</u> of R' with respect to S. Let $\varphi_S' : R' \to S^{-1}R'$ be given by $\varphi_S'(x) = \frac{x}{1}$. Then φ_S' makes $S^{-1}R'$ into an R'-algebra, and φ_S' is an R-algebra map.

Elts of S become units in $S^{-1}R'$. Moreover, we have a UMP: $S^{-1}R'$ is a universal R'-algebra where elements of S are units.

If $\tau : R' \to R''$ is an R-alg. map then there is a comm. diagram of R-algebra maps

$$\begin{tikzcd} R' \arrow[r, "\tau"] \arrow[d, "\varphi_S"'] & R'' \arrow[d] \\ S^{-1}R' \arrow[r, "S^{-1}\tau"] & S^{-1}R'' \end{tikzcd}$$

, and $S^{-1}\tau$ is an $S^{-1}R$-algebra map.

Let $T \subset R'$ be the image of $S \in R$. Then T is mult. and $S^{-1}R' = T^{-1}R'$.

<u>Prop.</u> Let $R \supset S$ multiplicative, $T' \subset S^{-1}R$ mult. subset, $T = \varphi_S^{-1}(T')$. Let $S \subset T$. Then $(T')^{-1}S^{-1}R = T^{-1}R$.

<u>Pf.</u> Check $(T')^{-1}S^{-1}R$ has the UMP characterizing $T^{-1}R$. Clearly $\varphi_{T'}\varphi_S$ carries T into $((T')^{-1}S^{-1}R)^\times$. Now let $\psi : R \to R'$ carrying T into $(R')^\times$. Our job is to show that ψ factors uniquely through $(T')^{-1}S^{-1}R$.

First, ψ carries S into $R'^\times$ since $S \subset T$. So ψ factors through a unique map $\rho : S^{-1}R \to R'$. Now, given $r \in T'$, write $r = \frac{x}{s}$. Then $\frac{x}{1} = \frac{s}{1} r \in T'$ since $S \subset T$. So $x \in T$. Thus $\rho(r) = \psi(x)\rho(\frac{1}{s}) \in (R')^\times$. So ρ factors through a unique $\rho' : (T')^{-1}S^{-1}R \to R'$. Hence $\psi = \rho'\varphi_{T'}\varphi_S$ and ρ' is unique as required.

Cor. R a ring, $\mathfrak{p} \subset \mathfrak{q}$ prime ideals. Then $R_{\mathfrak{p}}$ is the localization of $R_{\mathfrak{q}}$ at the prime $\mathfrak{p} R_{\mathfrak{q}}$.

Pf. $S = R \setminus \mathfrak{q}$, $T' = R_{\mathfrak{q}} \setminus \mathfrak{p} R_{\mathfrak{q}}$. Let $T = \varphi_S^{-1}(T')$. Then $T = R \setminus \mathfrak{p}$ (by incl-pres. bijection above for prime ideals). So $S \subset T$ and we get the assertion. ($\varphi_S^{-1}(\mathfrak{p} R_{\mathfrak{q}}) = \mathfrak{p}$)

Prop. $R \supset S$, $S^{-1}R[X] = S^{-1}(R[X])$

Pf. From univ. property.

Cor. $R \supset S$ mult. subset, X a variable, $\mathfrak{p} \subset R[X]$ ideal, $R' = S^{-1}R$, $\varphi: R[X] \to R'[X]$ can. map. Then $\mathfrak{p}$ is prime and $\mathfrak{p} \cap S = \emptyset \Leftrightarrow \mathfrak{p} R'[X]$ is prime and $\mathfrak{p} = \varphi^{-1}(\mathfrak{p} R'[X])$

Pf. Follows from the above and bijection for prime ideals.

Ex. $R = \mathbb{C}[X_1, \dots, X_n]$, f irred. polyn., $\mathfrak{p} = \langle f \rangle$. Then $R_{\langle \mathfrak{p} \rangle} = R_{\mathfrak{p}} =$ rat. functions which are generically regular at $f = 0$, i.e. ones whose denom is not div. by f. Then $\mathfrak{p} R_{\mathfrak{p}}$ – functions like that having f in the numerator. So $R_{\mathfrak{p}} / \mathfrak{p} R_{\mathfrak{p}} = \mathbb{C}(S)$, where S is the surface $f = 0$.

Ex. $\mathbb{C}[x,y]_{\langle x,y \rangle}$ = rational fns of x, y defined at 0

$\mathbb{C}[x,y]_{\langle x,y \rangle} \subset \mathbb{C}[x,y]_{\langle y \rangle}$ = rat fns defined generically at $y = 0$.

and $\mathbb{C}[x,y]_{\langle x,y \rangle} = \left(\mathbb{C}[x,y]_{\langle x,y \rangle} \right)_{\langle y \rangle}$.