Lectures and problems in representation theory

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1 Introduction

What is representation theory? To say it in one sentence, it is an exciting area of mathematics which studies representations of associative algebras. Representation theory has a wide variety of applications, ranging from physics (elementary particles) and chemistry (atoms, molecules) to probability (card shuffles) and number theory (Fermat's last theorem).

Representation theory was born in 1896 in the work of the German mathematician F. G. Frobenius. This work was triggered by a letter to Frobenius by R. Dedekind. In this letter Dedekind made the following observation: take the multiplication table of a finite group G and turn it into a matrix X_G by replacing every entry g of this table by a variable x_g. Then the determinant of X_G factors into a product of irreducible polynomials in x_g, each of which occurs with multiplicity equal to its degree. Dedekind checked this surprising fact in a few special cases, but could not prove it in general. So he gave this problem to Frobenius. In order to find a solution of this problem (which we will explain below), Frobenius created representation theory of finite groups.

The general content of representation theory can be very briefly summarized as follows.

An **associative algebra** over a field k is a vector space A over k equipped with an associative bilinear multiplication $a, b \to ab$, $a, b \in A$. We will always consider associative algebras with unit, i.e., with an element 1 such that $1 \cdot a = a \cdot 1 = a$ for all $a \in A$. A basic example of an associative algebra is the algebra $\mathrm{End}V$ of linear operators from a vector space V to itself. Other important examples include algebras defined by generators and relations, such as group algebras and univeral enveloping algebras of Lie algebras.

A **representation** of an associative algebra A (also called a left A-module) is a vector space V equipped with a homomorphism $\rho : A \to \mathrm{End}V$, i.e., a linear map preserving the multiplication and unit.

A **subrepresentation** of a representation V is a subspace $U \subset V$ which is invariant under all operators $\rho(a)$, $a \in A$. Also, if V_1, V_2 are two representations of A then the **direct sum** $V_1 \oplus V_2$ has an obvious structure of a representation of A.

A representation V of A is said to be **irreducible** if its only subrepresentations are 0 and V itself, and **indecomposable** if it cannot be written as a direct sum of two nonzero subrepresentations. Obviously, irreducible implies indecomposable, but not vice versa.

Typical problems of representation theory are as follows:

1. Classify irreducible representations of a given algebra A.
2. Classify indecomposable representations of A.
3. Do 1 and 2 restricting to finite dimensional representations.

As mentioned above, the algebra A is often given to us by generators and relations. For example, the universal enveloping algebra U of the Lie algebra $sl(2)$ is generated by h, e, f with defining relations

$$he - eh = 2e, \quad hf - fh = -2f, \quad ef - fe = h. \tag{1}$$

This means that the problem of finding, say, N-dimensional representations of A reduces to solving a bunch of nonlinear algebraic equations with respect to a bunch of unknown N by N matrices, for example system (1) with respect to unknown matrices h, e, f.

It is really striking that such, at first glance hopelessly complicated, systems of equations can in fact be solved completely by methods of representation theory! For example, we will prove the following theorem.

Theorem 1.1. *Let $k = \mathbb{C}$ be the field of complex numbers. Then:*

(i) The algebra U has exactly one irreducible representation V_d of each dimension, up to equivalence; this representation is realized in the space of homogeneous polynomials of two variables x, y of degree $d-1$, and defined by the formulas

$$\rho(h) = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}, \quad \rho(e) = x\frac{\partial}{\partial y}, \quad \rho(f) = y\frac{\partial}{\partial x}.$$

(ii) Any indecomposable finite dimensional representation of U is irreducible. That is, any finite dimensional representation of U is a direct sum of irreducible representations.

As another example consider the representation theory of quivers.

A **quiver** is a finite oriented graph Q. A **representation** of Q over a field k is an assignment of a k-vector space V_i to every vertex i of Q, and of a linear operator $A_h : V_i \to V_j$ to every directed edge h going from i to j. We will show that a representation of a quiver Q is the same thing as a representation of a certain algebra P_Q called the path algebra of Q. Thus one may ask: what are indecomposable finite dimensional representations of Q?

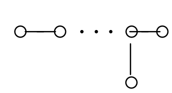
More specifically, let us say that Q is **finite** if it has finitely many indecomposable representations.

We will prove the following striking theorem, proved by P. Gabriel about 30 years ago:

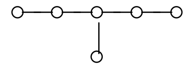
Theorem 1.2. *The finiteness property of Q does not depend on the orientation of edges. The graphs that yield finite quivers are given by the following list:*

- *A_n :*

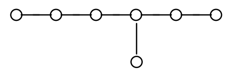
- *D_n:*



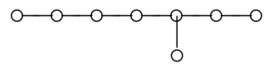
- E_6 :



- E_7 :



- E_8 :



The graphs listed in the theorem are called (simply laced) **Dynkin diagrams**. These graphs arise in a multitude of classification problems in mathematics, such as classification of simple Lie algebras, singularities, platonic solids, reflection groups, etc. In fact, if we needed to make contact with an alien civilization and show them how sophisticated our civilization is, perhaps showing them Dynkin diagrams would be the best choice!

As a final example consider the representation theory of finite groups, which is one of the most fascinating chapters of representation theory. In this theory, one considers representations of the group algebra $A = \mathbb{C}[G]$ of a finite group G – the algebra with basis $a_g, g \in G$ and multiplication law $a_g a_h = a_{gh}$. We will show that any finite dimensional representation of A is a direct sum of irreducible representations, i.e. the notions of an irreducible and indecomposable representation are the same for A (Maschke's theorem). Another striking result discussed below is the Frobenius' divisibility theorem: the dimension of any irreducible representation of A divides the order of G. Finally, we will show how to use representation theory of finite groups to prove Burnside's theorem: any finite group of order $p^a q^b$, where p, q are primes, is solvable. Note that this theorem does not mention representations, which are used only in its proof; a purely group-theoretical proof of this theorem (not using representations) exists but is much more difficult!

This text is based on a mini-course given by Pavel Etingof at the 2004 Clay Mathematics Institute Research Academy. The remaining authors, who were participants of the Academy, improved and extended the initial lecture notes, and added solutions of homework problems.

The goal of the text is not to provide a systematic introduction to representation theory, but rather to convey to the reader the spirit of this fascinating subject, and to highlight its beauty by discussing a few striking results. In other words, the authors would like to share with the reader the fun they had during the days of the Academy! [1].

The text contains many problems, whose solutions are given in the last section. These problems were given as homework during the CMI Research Academy. Sometimes they are designed to illustrate a notion or result from the main text, but often contain new material. The problems are a very important part of the text, and we recommend the reader to try to solve them after reading the appropriate portions.

The only serious prerequisite for reading this text is a good familiarity with linear algebra, and some proficiency in basic abstract algebra (groups, fields, polynomials etc.) The necessary material is discussed in the first seven chapters of Artin's "Algebra" textbook. Perhaps the only basic notion from linear algebra we'll need which is not contained in standard texts is that of tensor product of

[1]For more about the subject, we recommend the reader the excellent textbook of Fulton and Harris "Representation theory"

vector spaces; we recommend the reader to solve the preparatory problems below to attain a better familiarity with this notion.

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1.1 Preparatory problems on tensor products

Recall that the tensor product $V \otimes W$ of vector spaces V and W over a field k is the quotient of the space $V * W$ whose basis is given by formal symbols $v \otimes w$, $v \in V$, $w \in W$, by the subspace spanned by the elements

$$(v_1+v_2)\otimes w - v_1\otimes w - v_2\otimes w, v\otimes(w_1+w_2) - v\otimes w_1 - v\otimes w_2, (av)\otimes w - a(v\otimes w), v\otimes(aw) - a(v\otimes w),$$

where $v \in V, w \in W, a \in k$.

Problem 1.3. *(a) Let U be any k-vector space. Construct a natural bijection between bilinear maps $V \times W \to U$ and linear maps $V \otimes W \to U$.*

(b) Show that if $\{v_i\}$ is a basis of V and $\{w_j\}$ is a basis of W then $\{v_i \otimes w_j\}$ is a basis of $V \otimes W$.

(c) Construct a natural isomorphism $V^* \otimes W \to \mathrm{Hom}(V, W)$ in the case when V is finite dimensional.

(d) Let V be a vector space over $\mathbb{C}$. Let S^nV be the subspace of $V^{\otimes n}$ (n-fold tensor product of V) which consists of the tensors that are symmetric under permutation of components. Let Λ^nV be the subspace of $V^{\otimes n}$ which consists of the tensors which are antisymmetric, i.e., $s_{ij}T = -T$, where s_{ij} is the permutation of i and j. (These spaces are called the n-th symmetric, respectively exterior, power of V). If $\{v_i\}$ is a basis of V, can you construct a basis of S^nV, Λ^nV? If $\dim V = m$, what are their dimensions?

(e) Let $A : V \to W$ be a linear operator. Then we have $A^{\otimes n} : V^{\otimes n} \to W^{\otimes n}$, and its restrictions $S^nA : S^nV \to S^nW$, $\Lambda^nA : \Lambda^nV \to \Lambda^nW$. Suppose $V = W$ and has dimension N, and assume that the eigenvalues of A are $\lambda_1, ..., \lambda_N$. Find $Tr(S^nA), Tr(\Lambda^nA)$. In particular, show that $Tr(\Lambda^NA) = det(A)$.

Problem 1.4. *Let J_N be the linear operator on $\mathbb{C}^N$ given in the standard basis by the formula $J_Ne_i = e_{i-1}$ for $i > 1$, $J_Ne_1 = 0$. Thus J_N is a Jordan block of size N.*

Find the Jordan normal form of the operator $B = J_N \otimes 1_M + 1_N \otimes J_M$ on $\mathbb{C}^N \otimes \mathbb{C}^M$, where 1_L denotes the identity operator on $\mathbb{C}^L$.

Hint. Compute dimensions of kernels of B^j for all j.

Problem 1.5. Hilbert's problem. *It is known that if A and B are two polygons of the same area then A can be cut by finitely many straight cuts into pieces from which one can make B. David Hilbert asked in 1900 whether it is true for polyhedra in 3 dimensions. In particular, is it true for a cube and a regular tetrahedron of the same volume?*

The answer is "no", as was found by Dehn in 1901. The proof is very beautiful. Namely, to any polyhedron A let us attach its "Dehn invariant" $D(A)$ in

$V = \mathbb{R} \otimes (\mathbb{R}/\mathbb{Q})$ *(the tensor product of $\mathbb{Q}$-vector spaces). Namely,*

$$D(A) = \sum_a l(a) \otimes \frac{\beta(a)}{\pi},$$

where a runs over edges of A and $l(a), \beta(a)$ are the length of a and the angle at a.

(a) Show that if you cut A into B and C by a straight cut, then $D(A) = D(B) + D(C)$.

(b) Show that $\alpha = \arccos(1/3)/\pi$ is not a rational number.

Hint. Let $p_n, n \geq 1$ be the sequence defined by the recursive equation $p_{n+1} = p_n^2 - 2$, $p_0 = 2/3$. Show that $p_n = 2\cos 2^n \pi\alpha$. On the other hand, show that p_n must take infinitely many different values. From this, derive that α cannot be rational.

(c) Using (a) and (b), show that the answer to Hilbert's question is negative. (Compute the Dehn invariant of the regular tetrahedron and the cube).

2 Basic notions of representation theory

2.1 Algebras

Let k be a field. Unless stated otherwise, we will assume that k is algebraically closed, i.e. any nonconstant polynomial with coefficients in k has a root in k. The main example is the field of complex numbers $\mathbb{C}$, but we will also consider fields of characteristic p, such as the algebraic closure $\overline{\mathbb{F}}_p$ of the finite field $\mathbb{F}_p$ of p elements.

Definition 2.1. An associative algebra over k is a vector space A over k together with a bilinear map $A \times A \to A$, $(a, b) \mapsto ab$, such that $(ab)c = a(bc)$.

Definition 2.2. A unit in an associative algebra A is an element $1 \in A$ such that $1a = a1 = a$.

Proposition 2.3. *If a unit exists, it is unique.*

Proof. Let $1, 1'$ be two units. Then $1 = 11' = 1'$. ☐

From now on, by an algebra A we will mean an associative algebra with a unit. We will also assume that $A \neq 0$.

Example 2.4. Here are some examples of algebras over k:

1. $A = k$.

2. $A = k[x_1, ..., x_n]$ – the algebra of polynomials in variables $x_1, ..., x_n$.

3. $A = \text{End}V$ – the algebra of endomorphisms of a vector space V over k (i.e., linear maps from V to itself). The multiplication is given by composition of operators.

4. The free algebra $A = k\langle x_1, ..., x_n\rangle$. The basis of this algebra consists of words in letters $x_1, ..., x_n$, and multiplication is simply concatenation of words.

5. The group algebra $A = k[G]$ of a group G. Its basis is $\{a_g, g \in G\}$, with multiplication law $a_g a_h = a_{gh}$.

Definition 2.5. An algebra A is commutative if $ab = ba$ for all $a, b \in A$.

For instance, in the above examples, A is commutative in cases 1 and 2, but not commutative in cases 3 (if $\dim V > 1$), and 4 (if $n > 1$). In case 5, A is commutative if and only if G is commutative.

Definition 2.6. A homomorphism of algebra $f : A \to B$ is a linear map such that $f(xy) = f(x)f(y)$ for all $x, y \in A$, and $f(1) = 1$.

2.2 Representations

Definition 2.7. A representation of an algebra A (also called a left A-module) is a vector space V together with a homomorphism of algebras $\rho : A \to \mathrm{End}V$.

Similarly, a right A-module is a space V equipped with an antihomomorphism $\rho : A \to \mathrm{End}V$; i.e., ρ satisfies $\rho(ab) = \rho(b)\rho(a)$ and $\rho(1) = 1$.

The usual abbreviated notation for $\rho(a)v$ is av for a left module and va for the right module. Then the property that ρ is an (anti)homomorphism can be written as a kind of associativity law: $(ab)v = a(bv)$ for left modules, and $(va)b = v(ab)$ for right modules.

Example 2.8. 1. $V = 0$.

2. $V = A$, and $\rho : A \to \mathrm{End}A$ is defined as follows: $\rho(a)$ is the operator of left multiplication by a, so that $\rho(a)b = ab$ (the usual product). This representation is called the *regular* representation of A. Similarly, one can equip A with a structure of a right A-module by setting $\rho(a)b := ba$.

3. $A = k$. Then a representation of A is simply a vector space over k.

4. $A = k\langle x_1, ..., x_n\rangle$. Then a representation of A is just a vector space V over k with a collection of arbitrary linear operators $\rho(x_1), ..., \rho(x_n) : V \to V$ (explain why!).

Definition 2.9. A subrepresentation of a representation V of an algebra A is a subspace $W \subset V$ which is invariant under all operators $\rho(a) : V \to V$, $a \in A$.

For instance, 0 and V are always subrepresentations.

Definition 2.10. A representation $V \neq 0$ of A is irreducible (or simple) if the only subrepresentations of V are 0 and V.

Definition 2.11. Let V_1, V_2 be two representations over an algebra A. A homomorphism (or intertwining operator) $\phi : V_1 \to V_2$ is a linear operator which commutes with the action of A, i.e. $\phi(av) = a\phi(v)$ for any $v \in V_1$. A homomorphism ϕ is said to be an isomorphism of representations if it is an isomorphism of vector spaces.

Note that if a linear operator $\phi : V_1 \to V_2$ is an isomorphism of representations then so is the lienar operator $\phi^{-1} : V_2 \to V_1$ (check it!).

Two representations between which there exists an isomorphism are said to be isomorphic. For practical purposes, two isomorphic representations may be regarded as "the same", although there could be subtleties related to the fact that an isomorphism between two representations, when it exists, is not unique.

Definition 2.12. Let $V = V_1, V_2$ be representations of an algebra A. Then the space $V_1 \oplus V_2$ has an obvious structure of a representation of A, given by $a(v_1 \oplus v_2) = av_1 \oplus av_2$.

Definition 2.13. A representation V of an algebra A is said to be indecomposable if it is not isomorphic to a direct sum of two nonzero representations.

It is obvious that an irreducible representation is indecomposable. On the other hand, we will see below that the converse statement is false in general.

One of the main problems of representation theory is to classify irredicible and indecomposable representations of a given algebra up to isomorphism. This problem is usually hard and often can be solved only partially (say, for finite dimensional representations). Below we will see a number of examples in which this problem is partially or fully solved for specific algebras.

We will now prove our first result – Schur's lemma. Although it is very easy to prove, it is fundamental in the whole subject of representation theory.

Proposition 2.14. *(Schur's lemma) Let V_1, V_2 be irreducible representations of an algebra A over any field F. Let $\phi : V_1 \to V_2$ be a nonzero homomorphism of representations. Then ϕ is an isomorphism.*

Proof. The kernel K of ϕ is a subrepresentation of V_1. Since $\phi \neq 0$, this subrepresentation cannot be V_1. So by irreducibility of V_1 we have $K = 0$. The image I of ϕ is a subrepresentation of V_2. Since $\phi \neq 0$, this subrepresentation cannot be 0. So by irreducibility of V_2 we have $I = V_2$. Thus ϕ is an isomorphism. □

Corollary 2.15. *(Schur's lemma for algebraically closed fields) Let V be a finite dimensional irreducible representation of an algebra A over an algebraically closed field k, and $\phi : V \to V$ is an intertwining operator. Then $\phi = \lambda \cdot \mathrm{Id}$ (the scalar operator).*

Proof. Let λ be an eigenvalue of ϕ (a root of the characteristic polynomial of ϕ). It exists since k is an algebraically closed field. Then the operator $\phi - \lambda\mathrm{Id}$ is an intertwining operator $V \to V$, which is not an isomorphism (since its determinant is zero). Thus by Schur's lemma this operator is zero, hence the result. □

Corollary 2.16. *Let A be a commutative algebra. Then every irreducible finite dimensional representation V of A is 1-dimensional.*

Remark. Note that a 1-dimensional representation of any algebra is automatically irreducible.

Proof. For any element $a \in A$, the operator $\rho(a) : V \to V$ is an intertwining operator. Indeed,

$$\rho(a)\rho(b)v = \rho(ab)v = \rho(ba)v = \rho(b)\rho(a)v$$

(the second equality is true since the algebra is commutative). Thus, by Schur's lemma, $\rho(a)$ is a scalar operator for any $a \in A$. Hence every subspace of V is a subrepresentation. So 0 and V are the only subspaces of V. This means that $\dim V = 1$ (since $V \neq 0$). □

Example 2.17. 1. $A = k$. Since representations of A are simply vector spaces, $V = A$ is the only irreducible and the only indecomposable representation.

2. $A = k[x]$. Since this algebra is commutative, the irreducible representations of A are its 1-dimensional representations. As we discussed above, they are defined by a single operator $\rho(x)$. In the 1-dimensional case, this is just a number from k. So all the irreducible representations of A are $V_\lambda = k$, $\lambda \in k$, which the action of A defined by $\rho(x) = \lambda$. Clearly, these representations are pairwise non-isomorphic.

The classification of indecomposable representations is more interesting. To obtain it, recall that any linear operator on a finite dimensional vector space V can be brought to Jordan normal form. More specifically, recall that the Jordan block $J_{\lambda,n}$ is the operator on k^n which in the standard basis is given by the formulas $J_{\lambda,n}e_i = \lambda e_i + e_{i-1}$ for $i > 1$, and $J_{\lambda,n}e_1 = \lambda e_1$. Then for any linear operator $B : V \to V$ there exists a basis of V such that the matrix of B in this basis is a direct sum of Jordan blocks. This implies that all the indecomosable representations of A are $V_{\lambda,n} = k^n$, $\lambda \in k$, with $\rho(x) = J_{\lambda,n}$. The fact that these representations are indecomposable and pairwise non-isomorphic follows from the Jordan normal form theorem (which in particular says that the Jordan normal form of an operator is unique up to permutation of blocks).

This example shows that an indecomposable representation of an algebra need not be irreducible.

Problem 2.18. *Let V be a nonzero finite dimensional representation of an algebra A. Show that it has an irreducible subrepresentation. Then show by example that this does not always hold for infinite dimensional representations.*

Problem 2.19. *Let A be an algebra over an algebraically closed field k. The center $Z(A)$ of A is the set of all elements $z \in A$ which commute with all elements of A. For example, if A is commutative then $Z(A) = A$.*

(a) Show that if V is an irreducible finite dimensional representation of A then any element $z \in Z(A)$ acts in V by multiplication by some scalar $\chi_V(z)$. Show that $\chi_V : Z(A) \to k$ is a homomorphism. It is called the **central character** *of V.*

(b) Show that if V is an indecomposable finite dimensional representation of A then for any $z \in Z(A)$, the operator $\rho(z)$ by which z acts in V has only one eigenvalue $\chi_V(z)$, equal to the scalar by which z acts on some irreducible subrepresentation of V. Thus $\chi_V : Z(A) \to k$ is a homomorphism, which is again called the central character of V.

(c) Does $\rho(z)$ in (b) have to be a scalar operator?

Problem 2.20. *Let A be an associaive algebra, and V a representation of A. By $\mathrm{End}_A(V)$ one denotes the algebra of all morphisms of representations $V \to V$. Show that $\mathrm{End}_A(A) = A^{op}$, the algebra A with opposite multiplication.*

Problem 2.21. *Prove the following "Infinite dimensional Schur's lemma" (due to Dixmier): Let A be an algebra over $\mathbb{C}$ and V be an irreducible representation of A with at most countable basis. Then any homomorphism of representations $\phi : V \to V$ is a scalar operator.*

Hint. By the usual Schur's lemma, the alegbra $D := \mathrm{End}_A(V)$ is an algebra with division. Show that D is at most countably dimensional. Suppose ϕ is not a scalar, and consider the subfield $\mathbb{C}(\phi) \subset D$. Show that $\mathbb{C}(\phi)$ is a simple transcendental extension of $\mathbb{C}$. Derive from this that $\mathbb{C}(\phi)$ is uncountably dimensional and obtain a contradiction.

2.3 Ideals

A *left ideal* of an algebra A is a subspace $I \subseteq A$ such that $aI \subseteq I$ for all $a \in A$. Similarly, a *right ideal* of an algebra A is a subspace $I \subseteq A$ such that $Ia \subseteq I$ for all $a \in A$. A *two-sided ideal* is a subspace that is both a left and a right ideal.

Left ideals are the same as subrepresentations of the regular representation A. Right ideals are the same as subrepresentations of the regular representation of the opposite algebra A^{op}, in which the action of A is right multiplication.

Below are some examples of ideals:

- If A is any algebra, 0 and A are two-sided ideals. An algebra A is called *simple* if 0 and A are its only two-sided ideals.

- If $\phi : A \to B$ is a homomorphism of algebras, then $\ker \phi$ is a two-sided ideal of A.
- If S is any subset of an algebra A, then the two-sided ideal *generated* by S is denoted $\langle S \rangle$ and is the span of elements of the form asb, where $a, b \in A$ and $s \in S$. Similarly we can define $\langle S \rangle_\ell = \mathrm{span}\{as\}$ and $\langle S \rangle_r = \mathrm{span}\{sb\}$, the left, respectively right, ideal generated by S.

2.4 Quotients

Let A be an algebra and I a two-sided ideal in A. Then A/I is the set of (additive) cosets of I. Let $\pi : A \to A/I$ be the quotient map. We can define multiplication in A/I by $\pi(a) \cdot \pi(b) := \pi(ab)$. This is well-defined because if $\pi(a) = \pi(a')$ then

$$\pi(a'b) = \pi(ab + (a' - a)b) = \pi(ab) + \pi((a' - a)b) = \pi(ab)$$

because $(a' - a)b \in Ib \subseteq I = \ker \pi$, as I is a right ideal; similarly, if $\pi(b) = \pi(b')$ then

$$\pi(ab') = \pi(ab + a(b' - b)) = \pi(ab) + \pi(a(b' - b)) = \pi(ab)$$

because $a(b' - b) \in aI \subseteq I = \ker \pi$, as I is also a left ideal. Thus multiplication in A/I is well-defined, and A/I is an algebra.

Similarly, if V is a representation of A, and $W \subset V$ is a subrepresentation, then V/W is also a representation. Indeed, let $\pi : V \to V/W$ be the quotient map, and set $\rho_{V/W}(a)\pi(x) := \pi(\rho_V(a)x)$.

Above we noted the equivalence of left ideals of A and subrepresentations of the regular representation of A. Thus, if I is a left ideal in A, then A/I is a representation of A.

Problem 2.22. *Let $A = k[x_1, ..., x_n]$ and $I \neq A$ be any ideal in A containing all homogeneous polynomials of degree $\geq N$. Show that A/I is an indecomposable representation of A.*

Problem 2.23. *Let $V \neq 0$ be a representation of A. We say that a vector $v \in V$ is cyclic if it generates V, i.e., $Av = V$. A representation admitting a cyclic vector is said to be cyclic. Show that*

(a) V is irreducible if and only if all nonzero vectors of V are cyclic.

(b) V is cyclic if and only if it is isomorphic to A/I, where I is a left ideal in A.

(c) Give an example of an indecomposable representation which is not cyclic.

Hint. Let $A = \mathbb{C}[x, y]/I_2$, where I_2 is the ideal spanned by homogeneous polynomials of degree ≥ 2 (so A has a basis $1, x, y$). Let $V = A^*$ be the space of linear functionals on A, with the action of A given by $(\rho(a)f)(b) = f(ba)$. Show that V provides a required example.

2.5 Algebras defined by generators and relations

A representation V of A is said to be *generated* by a subset S of V if V is the span of $\{as \mid a \in A, s \in S\}$.

If $f_1, \ldots, f_m$ are elements of the free algebra $k\langle x_1, \ldots, x_n \rangle$, we say that the algebra $A := k\langle x_1, \ldots, x_n \rangle / \langle \{f_1, \ldots, f_m\} \rangle$ is *generated by $x_1, \ldots, x_n$ with defining relations $f_1 = 0, \ \ldots, \ f_m = 0$.*

2.6 Examples of algebras

Throughout the following examples G will denote a group, and k a field.

1. The group algebra $k[G]$, whose basis is $\{e_g \mid g \in G\}$, and where multiplication is defined by $e_g e_h = e_{gh}$. A *representation of a group G over a field k* is a homomorphism of groups $\rho : G \to GL(V)$, where V is some vector space over k. In fact, a representation of G over k is the "same thing" as a representation of $k[G]$.

2. The Weyl algebra, $k\langle x, y\rangle / \langle yx - xy - 1\rangle$. A basis for the Weyl algebra is $\{x^i y^j\}$ (show this).

   The space $\mathbb{C}[t]$ is a representation of the Weyl algebra over $\mathbb{C}$ with action given by $xf = tf$ and $yf = df/dt$ for all $f \in \mathbb{C}[t]$. Thus, the Weyl algebra over $\mathbb{C}$ is the algebra of polynomial differential operators.

   Definition. A representation $\rho : A \to \operatorname{End} V$ is *faithful* if ρ is injective.

   $\mathbb{C}[t]$ is a faithful representation of the Weyl algebra.

3. The q-Weyl algebra over k, generated by x_+, x_-, y_+, y_- with defining relations $y_+x_+ = qx_+y_+$ and $x_+x_- = x_-x_+ = y_+y_- = y_-y_+ = 1$. One customarily writes x_+ as x, x_- as x^{-1}, y_+ as y, and y_- as y^{-1}.

Problem 2.24. *Let A be the Weyl algebra, generated over an algebraically closed field k by two generators x, y with the relation $yx - xy = 1$.*

(a) If $\operatorname{char} k = 0$*, what are the finite dimensional representations of A? What are the two-sided ideals in A? .*

(b) Suppose for the rest of the problem that $\operatorname{char} k = p$*. What is the center of A?*

(c) Find all irreducible finite dimensional representations of A.

Problem 2.25. *Let q be a nonzero complex number, and A be the algebra over $\mathbb{C}$ generated by $X^{\pm 1}$ and $Y^{\pm 1}$ with defining relations $XX^{-1} = X^{-1}X = 1, YY^{-1} = Y^{-1}Y = 1$, and $XY = qYX$.*

(a) What is the center of A for different q? If q is not a root of unity, what are the two-sided ideals in A?

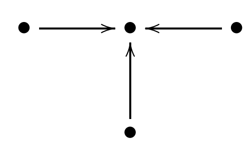
(b) For which q does this algebra have finite dimensional representations?

(c) Find all finite dimensional irreducible representations of A for such q.

2.7 Quivers

Definition 2.26. A **quiver** Q is a directed graph, possibly with self-loops and/or multiple arrows between two vertices.

Example 2.27.



We denote the set of vertices of the quiver Q as I, and the set of edges as E. For an edge $h \in E$, let h', h'' denote the source and target, respectively, of h.

$$\underset{h'}{\bullet} \xrightarrow{\ h\ } \underset{h''}{\bullet}$$

Definition 2.28. A representation of a quiver Q is an assignment to each vertex $i \in I$ of a vector space V_i and to each edge $h \in E$ of a linear map $x_h : V_{h'} \longrightarrow V_{h''}$.

It turns out that the theory of representations of quivers is part of the theory of representations of algebras in the sense that for each quiver Q, there exists a certain algebra P_Q, called the path algebra of Q, such that a representation of the quiver Q is "the same" as a representation of the algebra P_Q. We shall first define the path algebra of a quiver and then justify our claim that representations of these two objects are "the same".

Definition 2.29. The **path algebra** P_Q of a quiver Q is the algebra generated by p_i for $i \in I$ and a_h for $h \in E$ with the relations:

1. $p_i^2 = p_i$, $p_i p_j = 0$ for $i \neq j$
2. $a_h p_{h'} = a_h$, $a_h p_j = 0$ for $j \neq h'$
3. $p_{h''} a_h = a_h$, $p_i a_h = 0$ for $i \neq h''$

Remark 2.30. It is easy to see that $\sum_{i \in I} p_i = 1$, and that $a_{h_2} a_{h_1} = 0$ if $h_1'' \neq h_2'$.

This algebra is called the path algebra because a basis of P_Q is formed by elements a_π, where π is a path in Q (possibly of length 0). If $\pi = \overleftarrow{h_n \cdots h_2 h_1}$ (read right-to-left), then $a_\pi = a_{h_n} \cdots a_{h_2} a_{h_1}$. If π is a path of length 0, starting and ending at point i, then a_π is defined to be p_i. Clearly $a_{\pi_2} a_{\pi_1} = a_{\pi_2 \pi_1}$ (where $\pi_2 \pi_1$ is the concatenation of paths π_2 and π_1) if the final point of π_1 equals the initial point of π_2 and $a_{\pi_2} a_{\pi_1} = 0$ otherwise.

We now justify our statement that a representation of a quiver is the same as a representation of the path algebra of a quiver.

Let $\mathbf{V}$ be a representation of the path algebra P_Q. From this representation of the algebra P_Q, we can construct a representation of Q as follows: let $V_i = p_i \mathbf{V}$ and let $x_h = a_h|_{p_{h'}\mathbf{V}} : p_{h'}\mathbf{V} \longrightarrow p_{h''}\mathbf{V}$.

Similarly, let (V_i, x_h) be a representation of a quiver Q. From this representation, we can construct a representation of the path algebra P_Q: let $\mathbf{V} = \bigoplus_i V_i$, let $p_i : \mathbf{V} \to V_i \to \mathbf{V}$ be the projection onto V_i, and let $a_h = i_{h''} \circ x_h \circ p_{h'} : \mathbf{V} \to V_{h'} \to V_{h''} \to \mathbf{V}$ where $i_{h''} : V_{h''} \to \mathbf{V}$ is the inclusion map.

It is clear that the above assignments $\mathbf{V} \mapsto (p_i \mathbf{V})$ and $(V_i) \mapsto \bigoplus_i V_i$ are inverses of each other. Thus, we have a bijection between isomorphism classes of representations of the algebra P_Q and of the quiver Q.

Remark 2.31. In practice, it is generally easier to consider a representation of a quiver as in Definition 2.28. The above serves to show, as stated before, that the theory of representations of quivers is a part of the larger theory of representations of algebras.

We lastly define several previous concepts in the context of quivers representations.

Definition 2.32. A subrepresentation of a representation (V_i, x_h) of a quiver Q is a representation (W_i, x_h') where $W_i \subseteq V_i$ for all $i \in I$ and where $x_h(W_{h'}) \subseteq W_{h''}$ and $x_h' = x_h|_{W_{h'}} : W_{h'} \longrightarrow W_{h''}$ for all $h \in E$.

Definition 2.33. The direct sum of two representations (V_i, x_h) and (W_i, y_h) is the representation $(V_i \oplus W_i, x_h \oplus y_h)$.

As with representations of algebras, a representation (V_i) of a quiver Q is said to be irreducible if its only subrepresentations are (0) and (V_i) itself, and indecomposable if it is not isomorphic to a direct sum of two nonzero representations.

Definition 2.34. Let (V_i, x_h) and (W_i, y_h) be representations of the quiver Q. A homomorphism $\varphi : (V_i) \longrightarrow (W_i)$ of quiver representations is a collection of maps $\varphi_i : V_i \longrightarrow W_i$ such that $y_h \circ \varphi_{h'} = \varphi_{h''} \circ x_h$ for all $h \in E$.

Problem 2.35. *Let A be a $\mathbb{Z}_+$-graded algebra, i.e., $A = \oplus_{n \geq 0} A[n]$, and $A[n] \cdot A[m] \subset A[n+m]$. If $A[n]$ is finite dimensional, it is useful to consider the Hilbert series $h_A(t) = \sum \dim A[n] t^n$ (the generating function of dimensions of $A[n]$). Often this series converges to a rational function, and the answer is written in the form of such function. For example, if $A = k[x]$ and $deg(x^n) = n$ then*

$$h_A(t) = 1 + t + t^2 + ... + t^n + ... = \frac{1}{1-t}$$

Find the Hilbert series of:

(a) $A = k[x_1, ..., x_m]$ (where the grading is by degree of polynomials);

(b) $A = k < x_1, ..., x_m >$ (the grading is by length of words);

(c) A is the exterior algebra $\wedge_k[x_1, ..., x_m]$, generated by $x_1, ..., x_m$ with the defining relations $x_i x_j + x_j x_i = 0$ for all i, j (grading is by degree).

(d) A is the path algebra P_Q of a quiver Q as defined in the lectures.

Hint. The closed answer is written in terms of the adjacency matrix M_Q of Q.

2.8 Lie algebras

Let $\mathfrak{g}$ be a vector space over a field k, and let $[\,,] : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$ be a skew-symmetric bilinear map. (So $[a, b] = -[b, a]$.) If k is of characteristic 2, we also require that $[x, x] = 0$ for all x (a requirement equivalent to $[a, b] = -[b, a]$ in fields of other characteristics).

Definition 2.36. $(\mathfrak{g}, [\,,])$ is a **Lie algebra** if $[\,,]$ satisfies the Jacobi identity

$$\big[[a, b], c\big] + \big[[b, c], a\big] + \big[[c, a], b\big] = 0. \tag{2}$$

Example 2.37. Some examples of Lie algebras are:

1. $\mathbb{R}^3$ with $[u, v] = u \times v$, the cross-product of u and v
2. Any space g with $[\,,] = 0$ (abelian Lie algebra)
3. Any associative algebra A with $[a, b] = ab - ba$
4. Any subspace U of an associative algebra A such that $[a, b] \in U$ for all $a, b \in U$
5. $sl(n)$, the set of $n \times n$ matrices with trace 0
   For example, $sl(2)$ has the basis

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

   with relations $[e, f] = h$, $[h, f] = -2f$, $[h, e] = 2e$.

6. The Heisenberg Lie algebra $\mathcal{H}$ of matrices $\left(\begin{smallmatrix}0&*&*\\0&0&*\\0&0&0\end{smallmatrix}\right)$
   It has the basis
$$x=\begin{pmatrix}0&0&0\\0&0&1\\0&0&0\end{pmatrix}\qquad y=\begin{pmatrix}0&1&0\\0&0&0\\0&0&0\end{pmatrix}\qquad c=\begin{pmatrix}0&0&1\\0&0&0\\0&0&0\end{pmatrix}$$
   with relations $[y,x]=c$ and $[y,c]=[x,c]=0$.

7. The algebra $\mathrm{aff}(1)$ of matrices $\left(\begin{smallmatrix}*&*\\0&0\end{smallmatrix}\right)$
   Its basis consists of $X=\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)$ and $Y=\left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right)$, with $[X,Y]=Y$.

8. $so(n)$, the space of skew-symmetric $n\times n$ matrices, with $[a,b]=ab-ba$

Definition 2.38. Let $\mathfrak{g}_1,\mathfrak{g}_2$ be Lie algebras. A homomorphism $\varphi:\mathfrak{g}_1\longrightarrow\mathfrak{g}_2$ of Lie algebras is a linear map such that $\varphi([a,b])=[\varphi(a),\varphi(b)]$.

Definition 2.39. A representation of a Lie algebra $\mathfrak{g}$ is a vector space V with a homomorphism of Lie algebras $\rho:\mathfrak{g}\longrightarrow \mathrm{End}\,V$.

Example 2.40. Some examples of representations of Lie algebras are:

1. V=0

2. Any vector space V with $\rho=0$

3. Adjoint representation $V=\mathfrak{g}$ with $\rho(a)(b)=[a,b]\overset{def}{=}ab-ba$
   That this is a representation follows from Equation (2).

It turns out that a representation of a Lie algebra $\mathfrak{g}$ is the same as a representation of a certain associative algebra $\mathcal{U}(\mathfrak{g})$. Thus, as with quivers, we can view the theory of representations of Lie algebras as part of the theory of representations of associative algebras.

Definition 2.41. Let $\mathfrak{g}$ be a Lie algebra with basis x_i and $[\,,]$ defined by $[x_i,x_j]=\sum_k c_{ij}^k x_k$. The **universal enveloping algebra** $\mathcal{U}(\mathfrak{g})$ is the associative algebra generated by the x_i's with the relations $x_ix_j-x_jx_i=\sum_k c_{ij}^k x_k$.

Example 2.42. The associative algebra $\mathcal{U}(sl(2))$ is the algebra generated by e, f, h with relations
$$he-eh=2e\qquad hf-fh=-2f\qquad ef-fe=h.$$

Example 2.43. The algebra $\mathcal{U}(\mathcal{H})$, where $\mathcal{H}$ is the Heisenberg Lie algebra of Example 2.37.6, is the algebra generated by x, y, c with the relations
$$yx-xy=c\qquad yc-cy=0\qquad xc-cx=0.$$
The Weyl algebra is the quotient of $\mathcal{U}(\mathcal{H})$ by the relation $c=1$.

Finally, let us define the important notion of tensor product of representations.

Definition 2.44. The tensor product of two representations V,W of a Lie algebra $\mathfrak{g}$ is the space $V\otimes W$ with $\rho_{V\otimes W}(x)=\rho_V(x)\otimes Id+Id\otimes\rho_W(x)$.

It is easy to check that this is indeed a representation.

Problem 2.45. Representations of $sl(2)$. *According to the above, a representation of $sl(2)$ is just a vector space V with a triple of operators E, F, H such that $HE - EH = 2E, HF - FH = -2F, EF - FE = H$ (the corresponding map ρ is given by $\rho(e) = E, \rho(f) = F$, $\rho(h) = H$.*

Let V be a finite dimensional representation of $sl(2)$ (the ground field in this problem is $\mathbb{C}$).

(a) Take eigenvalues of H and pick one with the biggest real part. Call it λ. Let $\bar{V}(\lambda)$ be the generalized eigenspace corresponding to λ. Show that $E|_{\bar{V}(\lambda)} = 0$.

(b) Let W be any representation of $sl(2)$ and $w \in W$ be a nonzero vector such that $Ew = 0$. For any $k > 0$ find a polynomial $P_k(x)$ of degree k such that $E^k F^k w = P_k(H)w$. (First compute $EF^k w$, then use induction in k).

(c) Let $v \in \bar{V}(\lambda)$ be a generalized eigenvector of H with eigenvalue λ. Show that there exists $N > 0$ such that $F^N v = 0$.

(d) Show that H is diagonalizable on $\bar{V}(\lambda)$. (Take N to be such that $F^N = 0$ on $\bar{V}(\lambda)$, and compute $E^N F^N v$, $v \in \bar{V}(\lambda)$, by (b). Use the fact that $P_k(x)$ does not have multiple roots).

(e) Let N_v be the smallest N satisfying (c). Show that $\lambda = N_v - 1$.

(f) Show that for each $N > 0$, there exists a unique up to isomorphism irreducible representation of sl(2) of dimension N. Compute the matrices E, F, H in this representation using a convenient basis. (For V finite dimensional irreducible take λ as in (a) and $v \in V(\lambda)$ an eigenvector of H. Show that $v, Fv, ..., F^\lambda v$ is a basis of V, and compute matrices of all operators in this basis.)

Denote the $\lambda+1$-dimensional irreducible representation from (f) by V_λ. Below you will show that any finite dimensional representation is a direct sum of V_λ.

(g) Show that the operator $C = EF + FE + H^2/2$ (the so-called Casimir operator) commutes with E, F, H and equals $\frac{\lambda(\lambda+2)}{2} Id$ on V_λ.

Now it will be easy to prove the direct sum decomposition. Assume the contrary, and let V be a representation of the smallest dimension, which is not a direct sum of smaller representations.

(h) Show that C has only one eigenvalue on V, namely $\frac{\lambda(\lambda+2)}{2}$ for some nonnegative integer λ. (use that the generalized eigenspace decomposition of C must be a decomposition of representations).

(i) Show that V has a subrepresentation $W = V_\lambda$ such that $V/W = nV_\lambda$ for some n (use (h) and the fact that V is the smallest which cannot be decomposed).

(j) Deduce from (i) that the eigenspace $V(\lambda)$ of H is $n+1$-dimensional. If $v_1, ..., v_{n+1}$ is its basis, show that $F^j v_i$, $1 \le i \le n+1$, $0 \le j \le \lambda$ are linearly independent and therefore form a basis of V (establish that if $Fx = 0$ and $Hx = \mu x$ then $Cx = \frac{\mu(\mu-2)}{2} x$ and hence $\mu = -\lambda$).

(k) Define $W_i = span(v_i, Fv_i, ..., F^\lambda v_i)$. Show that V_i are subrepresentations of V and derive a contradiction with the fact that V cannot be decomposed.

(l) (Jacobson-Morozov Lemma) Let V be a finite dimensional complex vector space and $A : V \to V$ a nilpotent operator. Show that there exists a unique, up to an isomorphism, representation of $sl(2)$ on V such that $E = A$. (Use the classification of the representations and Jordan normal form theorem)

(m) (Clebsch-Gordan decomposition) Find the decomposition into irreducibles of the representation $V_\lambda \otimes V_\mu$ of $sl(2)$.

Hint. For a finite dimensional representation V of $sl(2)$ it is useful to introduce the character $\chi_V(x) = Tr(e^{xH})$, $x \in \mathbb{C}$. Show that $\chi_{V \oplus W}(x) = \chi_V(x) + \chi_W(x)$ and $\chi_{V \otimes W}(x) = \chi_V(x)\chi_W(x)$.

Then compute the character of V_λ and of $V_\lambda \otimes V_\mu$ and derive the decomposition. This decomposition is of fundamental importance in quantum mechanics.

(n) Let $V = \mathbb{C}^M \otimes \mathbb{C}^N$, and $A = J_M(0) \otimes Id_N + Id_M \otimes J_N(0)$. Find the Jordan normal form of A using (l),(m), and compare the answer with Problem 1.4.

Problem 2.46. *(***Lie's Theorem***) Recall that the commutant $K(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$ is the linear span of elements $[x, y]$, $x, y \in \mathfrak{g}$. This is an ideal in $\mathfrak{g}$ (i.e. it is a subrepresentation of the adjoint representation). A finite dimensional Lie algebra $\mathfrak{g}$ over a field k is said to be solvable if there exists n such that $K^n(\mathfrak{g}) = 0$. Prove the Lie theorem: if $k = \mathbb{C}$ and V is a finite dimensional irreducible representation of a solvable Lie algebra $\mathfrak{g}$ then V is 1-dimensional.*

Hint. Prove the result by induction in dimension. By the induction assumption, $K(\mathfrak{g})$ has a common eigenvector v in V, that is there is a linear function $\chi : K(\mathfrak{g}) \to \mathbb{C}$ such that $av = \chi(a)v$ for any $a \in K(\mathfrak{g})$. Show that $\mathfrak{g}$ preserves common eigenspaces of $K(\mathfrak{g})$ (for this you will need to show that $\chi([x, a]) = 0$ for $x \in \mathfrak{g}$ and $a \in K(\mathfrak{g})$. To prove this, consider the smallest vector subspace U containing v and invariant under x. This subspace is invariant under $K(\mathfrak{g})$ and any $a \in K(\mathfrak{g})$ acts with trace $\dim(U)\chi(a)$ in this subspace. In particular $0 = \mathrm{Tr}([x, a]) = \dim(U)\chi([x, a])$.).

Problem 2.47. *Classify irreducible representations of the two dimensional Lie algebra with basis X, Y and commutation relation $[X, Y] = Y$. Consider the cases of zero and positive characteristic. Is the Lie theorem true in positive characteristic?*

Problem 2.48. *(hard!) For any element x of a Lie algebra $\mathfrak{g}$ let $ad(x)$ denote the operator $\mathfrak{g} \to \mathfrak{g}, y \mapsto [x, y]$. Consider the Lie algebra $\mathfrak{g}_n$ generated by two elements x, y with the defining relations $ad(x)^2(y) = ad(y)^{n+1}(x) = 0$.*

(a) Show that the Lie algebras $\mathfrak{g}_1, \mathfrak{g}_2, \mathfrak{g}_3$ are finite dimensional and find their dimensions.

(b) (harder!) Show that the Lie algebra $\mathfrak{g}_4$ has infinite dimension. Construct explicitly a basis of this algebra.

3 General results of representation theory

3.1 The density theorem

Theorem 3.1. *(The density theorem) Let A be an algebra, and let $V_1, V_2, \ldots, V_r$ be pairwise non-isomorphic irreducible finite dimensional representations of A, with homomorphisms $\rho_i : A \longrightarrow \mathrm{End}\, V_i$. Then the homomorphism*

$$\rho_1 \oplus \rho_2 \oplus \cdots \oplus \rho_r : A \longrightarrow \bigoplus_{i=1}^{r} \mathrm{End}\, V_i$$

is surjective.

Proof. Let

$$V_{N_1,\ldots,N_r} = \underbrace{(V_1 \oplus \cdots \oplus V_1)}_{N_1 \text{ copies}} \oplus \underbrace{(V_2 \oplus \cdots \oplus V_2)}_{N_2 \text{ copies}} \oplus \cdots \oplus \underbrace{(V_r \oplus \cdots \oplus V_r)}_{N_r \text{ copies}}.$$

Let $p_{ij} : V_{N_1,\ldots,N_r} \longrightarrow V_i$ be the projection onto V_{ij}, the j^{th} copy of V_i.

We shall need the following two lemmas:

Lemma 3.2. *Let $W \subsetneq \mathbf{V} = V_{N_1,\ldots,N_r}$ be a subrepresentation. Then, there exists an automorphism α of $\mathbf{V}$ such that $p_{iN_i}(\alpha(W)) = 0$ for some i.*

Proof of Lemma 3.2. We induct on $N = \sum_{i=1}^r N_i$. The base case, $N = 0$, is clear. For the inductive step, first pick an irreducible nonzero subrepresentation Y of W (which clearly exists by Problem 2.18). As $Y \neq 0$, there exist some i, j such that $p_{ij}|_Y : Y \longrightarrow V_i$ is nonzero. Assume, without loss of generality, that $j = 1$. As $p_{i1}|_Y$ is nonzero, by Schur's lemma, $p_{i1}|_Y$ is an isomorphism, and $Y \cong V_i$. As Y can be isomorphic to only one V_l, $p_{lm}|_Y = 0$ for all $l \neq i, m$. As $p_{im}|_Y \circ (p_{i1}|_Y)^{-1} : V_i \to Y \to V_i$ is a homomorphism, by Schur's lemma for algebraically closed fields, $p_{im}|_Y \circ (p_{i1}|_Y)^{-1} = \lambda_m \cdot \mathrm{Id}$ and $p_{im}|_Y = \lambda_m \cdot p_{i1}|_Y$ for some scalar λ_m.

We now define an automorphism $\gamma : \mathbf{V} \longrightarrow \mathbf{V}$. For $v \in \mathbf{V}$, we write $v = (v_{lm})$ where $v_{lm} = p_{lm}(v) \in V_l$. We now let $\gamma(v) = (v'_{lm})$, where

$$\begin{aligned} v'_{lm} &= v_{lm} \text{ for } l \neq i, \\ v'_{i1} &= v_{i1}, \\ v'_{im} &= v_{im} - \lambda_m v_{i1} \text{ for } 2 \leq m \leq N_i. \end{aligned}$$

Clearly γ is an automorphism. Suppose that $v \in Y$. As $p_{lm}|_Y = 0$ for all $l \neq i, m$, $v'_{lm} = v_{lm} = p_{lm}(v) = 0$ for $l \neq i, m$. As $p_{im}|_Y = \lambda_m \cdot p_{i1}|_Y$, $v_{im} = \lambda_m v_{i1}$ and $v'_{im} = v_{im} - \lambda_m v_{i1} = 0$ for $m \neq 1$. Thus, $p_{lm}(\gamma(Y)) = v'_{lm} = 0$ unless $l = i$ and $m = 1$. Also, $p_{i1}(\gamma(Y)) \cong Y$.

Consider the map

$$\begin{aligned} \varphi : \mathbf{V} = V_{N_1,\ldots,N_i,\ldots,N_r} &\longrightarrow V_{N_1,\ldots,N_i-1,\ldots,N_r} \\ (v_{lm}) &\longmapsto (v_{lm} \text{ for } (l,m) \neq (i,1)) \end{aligned}$$

which has V_{i1} (the first copy of V_i) as its kernel. But this is also $\gamma(Y)$, so $\ker\varphi = \gamma(Y)$. As $\gamma(Y) \subseteq \gamma(W)$, $\ker \varphi|_{\gamma(W)} = \gamma(Y)$ and $\varphi(\gamma(W)) \cong \gamma(W)/\gamma(Y) \cong W/Y$.

By the inductive hypothesis, there exists some $\beta : V_{N_1,\ldots,N_i-1,\ldots,N_r} \longrightarrow V_{N_1,\ldots,N_i-1,\ldots,N_r}$ such that for some l, $p_{lN'_l}(\beta(\varphi(\gamma(W)))) = 0$, where $N'_l = N_l$ for $l \neq i$ and $N'_i = N_i - 1$. Define $\alpha = (\mathrm{Id}_{V_{i1}} \oplus \beta) \circ \gamma$ (where $\mathrm{Id}_{V_{i1}}$ is the identity on the first copy of V_i). As $\mathrm{Id}_{V_{i1}} \oplus \varphi = \mathrm{Id}_{\mathbf{V}}$, $\mathrm{Id}_{V_{i1}} \oplus \beta = (\mathrm{Id}_{V_{i1}} \oplus 0) + (\beta \circ \varphi)$. Thus, $p_{lN_l}(\alpha(W)) = p_{lN'_l}(\beta(\varphi(\gamma(W)))) = 0$ (as $(l, N_l) \neq (i, 1)$). This completes the inductive step. ☐

Lemma 3.3. *Let $W \subseteq V_{N_1,\ldots,N_r}$ be a subrepresentation. Then, $W \cong V_{M_1,\ldots,M_r}$ for some $M_i \leq N_i$.*

Proof of Lemma 3.3. We induct on $N = \sum_{i=1}^r N_i$. The base case, $N = 0$ is clear, as then $W = 0$. If $W = V_{N_1,\ldots,N_r}$, then we simply have $M_i = N_i$. Otherwise, by Lemma 3.2, there exists an automorphism α of $V_{N_1,\ldots,N_r}$ such that $p_{iN_i}(\alpha(W)) = 0$ for some i. Thus, $\alpha(W) \subseteq V_{N_1,\ldots,N_i-1,\ldots,N_r}$. By the inductive assumption, $W \cong \alpha(W) \cong V_{M_1,\ldots,M_r}$. ☐

Proof of Theorem 3.1. First, by replacing A with $A/\ker \rho_1 \oplus \rho_2 \oplus \cdots \oplus \rho_r$, we can assume, without loss of generality, that the map $\rho_1 \oplus \rho_2 \oplus \cdots \oplus \rho_r$ is injective. As A is now isomorphic to its image, we can also assume that $A \subseteq \mathrm{End}\, V_1 \oplus \cdots \oplus \mathrm{End}\, V_r$. Thus, A is a subrepresentation of $\mathrm{End}\, V_1 \oplus \cdots \oplus \mathrm{End}\, V_r$.

Let $d_i = \dim V_i$. As $\mathrm{End}\, V_i \cong \underbrace{V_i \oplus \cdots \oplus V_i}_{d_i \text{ copies}}$, we have $A \subseteq V_{d_1,\ldots,d_r}$. Thus, by Lemma 3.3, $A \cong V_{M_1,\ldots,M_r}$ for some $M_i \leq d_i$. Thus, $\dim A = \sum_{i=1}^r M_i d_i$.

Next, consider $\mathrm{End}_A(A)$. As the V_i's are pairwise non-isomorphic, by Schur's lemma, no copy of V_i in A can be mapped to a distinct V_j. Also, by Schur, $\mathrm{End}_A(V_i) = k$. Thus, $\mathrm{End}_A(A) \cong \bigoplus_i \mathrm{Mat}_{M_i}(k)$, so $\dim \mathrm{End}_A(A) = \sum_{i=1}^r M_i^2$. By Problem 2.20, $\mathrm{End}_A(A) \cong A$, so $\sum_{i=1}^r M_i^2 = \dim A = \sum_{i=1}^r M_i d_i$. Thus, $\sum_{i=1}^r M_i(d_i - M_i) = 0$. As $M_i \le d_i$, $d_i - M_i \ge 0$. Next, as $1 \in A$, the map $\rho_i : A \longrightarrow \mathrm{End}\, V_i$ is nontrivial. As $\mathrm{End}\, V_i$ is a direct sum of copies of V_i, A must contain a copy of V_i. Thus $M_i > 0$, and we must have $d_i - M_i = 0$ for all i, so $M_i = d_i$, and $A = \bigoplus_{i=1}^r \mathrm{End}\, V_i$. ☐

3.2 Representations of matrix algebras

In this section we consider representations of algebras $A = \bigoplus_i \mathrm{Mat}_{d_i}(k)$.

Theorem 3.4. *Let $A = \bigoplus_{i=1}^r \mathrm{Mat}_{d_i}(k)$. Then the irreducible representations of A are $V_1 = k^{d_1}, \ldots, V_r = k^{d_r}$, and any finite dimensional representation of A is a direct sum of copies of $V_1, \ldots, V_r$.*

In order to prove Theorem 3.4, we shall need the notion of a dual representation.

Definition 3.5. (Dual representation) Let V be a representation of A. Then the dual space V^* is a right A-module (or equivalently, a representation of A^{op}) with the action

$$f \cdot a = (v \longmapsto f(av)),$$

as $(f \cdot (ab))(v) = f((ab)v) = f(a(bv)) = (f \cdot a)(bv) = ((f \cdot a) \cdot b)(v)$.

Proof of Theorem 3.4. First, the given representations are clearly irreducible, as for any $v, w \in V_i \setminus 0$, there exists $a \in A$ such that $av = w$. Next, let X be an n dimensional representation of A. Then, X^* is an n dimensional representation of A^{op}. But $(\mathrm{Mat}_{d_i}(k))^{\mathrm{op}} \cong \mathrm{Mat}_{d_i}(k)$ with isomorphism $\varphi(X) = X^T$, as $(AB)^T = B^T A^T$. Thus, $A \cong A^{\mathrm{op}}$ and X^* is an n dimensional representation of A. Define

$$\phi : \underbrace{A \oplus \cdots \oplus A}_{n \text{ copies}} \longrightarrow X^*$$

by

$$\phi(a_1, \ldots, a_n) = a_1 x_1^* + \cdots + a_n x_n^*$$

where $\{x_i^*\}$ is a basis of X^*. ϕ is clearly surjective, as $k \subset A$. Thus, the dual map $\phi^* : X \longrightarrow A^{n*}$ is injective. But $A^{n*} \cong A^n$. Hence, $\mathrm{Im}\, \phi^* \cong X$ is a subrepresentation of A^n. Next, as $\mathrm{Mat}_{d_i}(k) = V_i^{d_i}$, $A = V_{d_1,\ldots,d_r}$ and $A^n = V_{nd_1,\ldots,nd_r}$. By Lemma 3.3, $X \cong V_{M_1,\ldots,M_r} = M_1 V_1 \oplus \cdots \oplus M_r V_r$ for some M_i as desired. ☐

3.3 Finite dimensional algebras

Definition 3.6. The **radical** I of a finite dimensional algebra A is the set of all elements of A which act by 0 in all irreducible representations of A. It is denoted $\mathrm{Rad}(A)$.

Proposition 3.7. *$\mathrm{Rad}(A)$ is a two-sided ideal.*

Proof. If $i \in I$, $a \in A$, $v \in V$, where V is any irreducible representation of A, then $aiv = a \cdot 0 = 0$ and $iav = iv' = 0$ where $v' = av$. ☐

Theorem 3.8. *A finite dimensional algebra A has only finitely many irreducible representations V_i up to isomorphism, these representations are finite dimensional, and*

$$A/I \cong \bigoplus_i \operatorname{End} V_i,$$

where $I = Rad(A)$.

Proof. First, for any irreducible representation V of A, and for any nonzero $v \in V$, $Av \subseteq V$ is a finite dimensional subrepresentation of V. (It is finite dimensional as A is finite dimensional.) As V is irreducible and $Av \neq 0$, $V = Av$ and V is finite dimensional.

Next, suppose that we had infinitely many non-isomorphic irreducible representations. Let $V_1, V_2, \ldots, V_r$ be any r nontrivial non-isomorphic irreducible representations, with $r > \dim A$. By Theorem 3.1, the homomorphism

$$\bigoplus_i \rho_i : A \longrightarrow \bigoplus_i \operatorname{End} V_i$$

is surjective. But this is impossible as $\sum \dim \operatorname{End} V_i \geq r > \dim A$. Thus, A has only finitely many non-isomorphic irreducible representations.

Next, let $V_1, V_2, \ldots, V_r$ be all non-isomorphic irreducible finite dimensional representations of A. By Theorem 3.1, the homomorphism

$$\bigoplus_i \rho_i : A \longrightarrow \bigoplus_i \operatorname{End} V_i$$

is surjective. The kernel of this map is exactly I. $\square$

Corollary 3.9. $\sum_i (\dim V_i)^2 \leq \dim A$, *where the V_i's are the irreducible representations of A.*

Proof. As $\dim \operatorname{End} V_i = (\dim V_i)^2$, Theorem 3.8 implies that $\dim A - \dim I = \sum_i \dim \operatorname{End} V_i = \sum_i (\dim V_i)^2$. As $\dim I \geq 0$, $\sum_i (\dim V_i)^2 \leq \dim A$. $\square$

Definition 3.10. A finite dimensional algebra A is said to be **semisimple** if $\operatorname{Rad}(A) = 0$.

Proposition 3.11. *For a finite dimensional algebra A over an algebraically closed field k, the following are equivalent:*

1. *A is semisimple*
2. $\sum_i (\dim V_i)^2 = \dim A$, *where the V_i's are the irreducible representations of A*
3. $A \cong \bigoplus_i \operatorname{Mat}_{d_i}(k)$ *for some d_i*
4. *Any finite dimensional representation of A is completely reducible (that is, isomorphic to a direct sum of irreducible representations)*
5. *A (as a vector space) is a completely reducible representation of A*

Proof. As $\dim A - \dim I = \sum_i (\dim V_i)^2$, clearly $\dim A = \sum_i (\dim V_i)^2$ if and only if $I = 0$. Thus, (1) $\Leftrightarrow$ (2).

Next, by Theorem 3.8, if $I = 0$, then clearly $A \cong \bigoplus_i \operatorname{Mat}_{d_i}(k)$ for $d_i = \dim V_i$. Thus, (1) $\Rightarrow$ (3). Conversely, if $A \cong \bigoplus_i \operatorname{Mat}_{d_i}(k)$, then $A \cong \bigoplus_i \operatorname{End} U_i$ for some U_i's with $\dim U_i = d_i$. Clearly each U_i is irreducible (as for any $u, u' \in U_i \setminus 0$, there exists $a \in A$ such that $au = u'$), and the U_i's are pairwise non-isomorphic representations. Thus, the U_i's form a subset of the irreducible representations V_i of A. Thus, $\dim A = \sum_i (\dim U_i)^2 \leq \sum_i (\dim V_i)^2 \leq \dim A$. Thus, (3) $\Rightarrow$ (2)($\Rightarrow$ (1)).

Next (3) $\Rightarrow$ (4) by Theorem 3.4. Clearly (4) $\Rightarrow$ (5). To see that (5) $\Rightarrow$ (3), let $A = \bigoplus_i n_i V_i$. Consider $\mathrm{End}_A(A)$ (endomorphisms of A as a representation of A). As the V_i's are pairwise non-isomorphic, by Schur's lemma, no copy of V_i in A can be mapped to a distinct V_j. Also, by Schur, $\mathrm{End}_A(V_i) = k$. Thus, $\mathrm{End}_A(A) \cong \bigoplus_i \mathrm{Mat}_{n_i}(k)$. But $\mathrm{End}_A(A) \cong A^{\mathrm{op}}$ by Problem 2.20, so $A^{\mathrm{op}} \cong \bigoplus_i \mathrm{Mat}_{n_i}(k)$. Thus, $A \cong (\bigoplus_i \mathrm{Mat}_{n_i}(k))^{\mathrm{op}} = \bigoplus_i (\mathrm{Mat}_{n_i}(k))^{\mathrm{op}}$. But $(\mathrm{Mat}_{n_i}(k))^{\mathrm{op}} \cong \mathrm{Mat}_{n_i}(k)$ with isomorphism $\varphi(X) = X^T$, as $(AB)^T = B^T A^T$. Thus, $A \cong \bigoplus_i \mathrm{Mat}_{n_i}(k)$. ∎

Let A be an algebra and V a finite-dimensional representation of A with action ρ. Then the *character* of V is the linear function $\chi_V : A \to k$ given by

$$\chi_V(a) = \mathrm{tr}|_V(\rho(a)).$$

If $[A, A]$ is the span of commutators $[x, y] := xy - yx$ over all $x, y \in A$, then $[A, A] \subseteq \ker \chi_V$. Thus, we may view the character as a mapping $\chi_V : A/[A, A] \to k$.

Theorem 3.12. *(1) Characters of irreducible finite-dimensional representations of A are linearly independent. (2) If A is a finite-dimensional semisimple algebra, then the characters form a basis of $(A/[A, A])^*$.*

Proof. (1) If $V_1, \ldots, V_r$ are nonisomorphic irreducible finite-dimensional representations of A, then $\rho_{V_1} \oplus \cdots \oplus \rho_{V_r} : A \to \mathrm{End}\ V_1 \oplus \cdots \oplus \mathrm{End}\ V_r$ is surjective by the density theorem, so $\chi_{V_1}, \ldots, \chi_{V_r}$ are linearly independent. (Indeed, if $\sum \lambda_i \chi_{V_i}(a) = 0$ for all $a \in A$, then $\sum \lambda_i \mathrm{Tr}(M_i) = 0$ for all $M_i \in \mathrm{End}_k V_i$. But each $\mathrm{tr}(M_i)$ can range independently over k, so it must be that $\lambda_1 = \cdots = \lambda_r = 0$.)

(2) First we prove that $[\mathrm{Mat}_d(k), \mathrm{Mat}_d(k)] = sl_d(k)$, the set of all matrices with trace 0. It is clear that $[\mathrm{Mat}_d(k), \mathrm{Mat}_d(k)] \subseteq sl_d(k)$. If we denote by E_{ij} the matrix with 1 in the ith row of the jth column and 0's everywhere else, we have $[E_{ij}, E_{jm}] = E_{im}$ for $i \neq m$, and $[E_{i,i+1}, E_{i+1,i}] = E_{ii} - E_{i+1,i+1}$. Now $\{E_{im}\} \cup \{E_{ii} - E_{i+1,i+1}\}$ forms a basis in $sl_d(k)$, and indeed $[\mathrm{Mat}_d(k), \mathrm{Mat}_d(k)] = sl_d(k)$, as claimed.

By semisimplicity, we can write $A = \mathrm{Mat}_{d_1} k \oplus \cdots \oplus \mathrm{Mat}_{d_r} k$. Then $[A, A] = sl_{d_1}(k) \oplus \cdots \oplus sl_{d_r}(k)$, and $A/[A, A] \cong k^r$. By the corollary to the density theorem, there are exactly r irreducible representations of A (isomorphic to $k^{d_1}, \ldots, k^{d_r}$, respectively), and therefore r linearly independent characters in the r-dimensional vector space $A/[A, A]$. Thus, the characters form a basis. ∎

3.4 Jordan-Holder and Krull-Schmidt theorems

To conclude the discussion of associative algebras, let us state two important theorems about their finite dimensional representations.

Let A be an algebra over an algebraically closed field k. Let V be a representation of A. A (finite) *filtration* of A is a sequence of subrepresentations $0 = V_0 \subset V_1 \subset \ldots \subset V_n = V$.

Theorem 3.13. *(Jordan-Holder theorem). Let V be a finite dimensional representation of A, and $0 = V_0 \subset V_1 \subset \ldots \subset V_n = V$, $0 = V'_0 \subset \ldots \subset V'_m = V$ be filtrations of V, such that the representations $W_i := V_i/V_{i-1}$ and $W'_i := V'_i/V'_{i-1}$ are irreducible for all i. Then $n = m$, and there exists a permutation σ of $1, \ldots, n$ such that $W_{\sigma(i)}$ is isomorphic to W'_i.*

Proof. **First proof** (for k of characteristic zero). Let $I \subset A$ be the annihilating ideal of V (i.e. the set of elements that act by zero in V). Replacing A with A/I, we may assume that A is finite dimensional. The character of V obviously equals the sum of characters of W_i, and also the sum of characters of W_i'. But by Theorem 3.12, the characters of irreducible representations are linearly independent, so the multiplicity of every irreducible representation W of A among W_i and among W_i' are the same. This implies the theorem.

Second proof (general). The proof is by induction on $\dim V$. The base of induction is clear, so let us prove the induction step. If $W_1 = W_1'$ (as subspaces), we are done, since by the induction assumption the theorem holds for V/W_1. So assume $W_1 \neq W_1'$. In this case $W_1 \cap W_1' = 0$ (as W_1, W_1' are irreducible), so we have an embedding $f : W_1 \oplus W_1' \to V$. Let $U = V/(W_1 \oplus W_1')$, and $0 = U_0 \subset U_1 \subset ... \subset U_p = U$ be a filtration of U with simple quotients $Z_i = U_i/U_{i-1}$. Then we see that:

1) V/W_1 has a filtration with successive quotients $W_1', Z_1, ..., Z_p$, and another filtration with successive quotients $W_2, ..., W_n$.

2) V/Y has a filtration with successive quotients $W_1, Z_1, ..., Z_p$, and another filtration with successive quotients $W_2', ..., W_n'$.

By the induction assumption, this means that the collection of irreducible modules with multiplicities $W_1, W_1', Z_1, ..., Z_p$ coincides on one hand with $W_1, ..., W_n$, and on the other hand, with $W_1', ..., W_m'$. We are done. □

Theorem 3.14. *(Krull-Schmidt theorem) Any finite dimensional representation of A can be uniquely (up to order of summands) decomposed into a direct sum of indecomposable representations.*

Proof. It is clear that a decomposition of V into a direct sum of indecomposable representation exists, so we just need to prove uniqueness. We will prove it by induction on $\dim V$. Let $V = V_1 \oplus ... \oplus V_m = V_1' \oplus ... \oplus V_n'$. Let $i_s : V_s \to V$, $i_s' : V_s' \to V$, $p_s : V \to V_s$, $p_s' : V \to V_s'$ be the natural maps associated to these decompositions. Let $\theta_s = p_1 i_s' p_s' i_1 : V_1 \to V_1$. We have $\sum_{s=1}^n \theta_s = 1$. Now we need the following lemma.

Lemma 3.15. *Let W be a finite dimensional indecomposable representation of A. Then*

(i) Any homomorphism $\theta : W \to W$ is either an isomorphism of nilpotent;

(ii) If $\theta_s : W \to W$, $s = 1, ..., n$ are nilpotent homomorphisms, then so is $\theta := \theta_1 + ... + \theta_n$.

Proof. (i) Generalized eigenspaces of θ are subrepresentations of V, and V is their direct sum. Thus, θ can have only one eigenvalue λ. If λ is zero, θ is nilpotent, otherwise it is an isomorphism.

(ii) The proof is by induction in n. The base is clear. To make the induction step ($n-1$ to n), assume that θ is not nilpotent. Then by (i) θ is an isomorphism, so $\sum_{i=1}^n \theta^{-1}\theta_i = 1$. The morphisms $\theta^{-1}\theta_i$ are not isomorphisms, so they are nilpotent. Thus $1 - theta^{-1}\theta_n = \theta^{-1}\theta_1 + ... + \theta^{-1}\theta_{n-1}$ is an isomorphism, which is a contradiction with the induction assumption. □

By the lemma, we find that for some s, θ_s must be an isomorphism; we may assume that $s = 1$. In this case, $V_1' = \mathrm{Im} p_1' i_1 \oplus \mathrm{Ker}(p_1 i_1')$, so since V_1' is indecomposable, we get that $f := p_1' i_1 : V_1 \to V_1'$ and $g := p_1 i_1' : V_1' \to V_1$ are isomorphisms.

Let $B = \oplus_{j>1} V_j$, $B' = \oplus_{j>1} V_j'$; then we have $V = V_1 \oplus B = V_1' \oplus B'$. Consider the map $h : B \to B'$ defined as a composition of the natural maps $B \to V \to B'$ attached to these decompositions. We claim that h is an isomorphism. To show this, it suffices to show that $\mathrm{Ker} h = 0$

(as h is a map between spaces of the same dimension). Assume that $v \in \mathrm{Ker} h \subset B$. Then $v \in V_1'$. On the other hand, the projection of v to V_1 is zero, so $gv = 1$. Since g is an isomorphism, we get $v = 0$, as desired.

Now by the induction assumption, $m = n$, and $V_j = V'_{\sigma(j)}$ for some permutation σ of $2, ..., n$. The theorem is proved. □

3.5 Problems

Problem 3.16. Extensions of representations.

Let A be an algebra over an algebraically closed field k, and V, W be a pair of representations of A. We would like to classify representations U of A such that V is a subrepresentation of U, and $U/V = W$. Of course, there is an obvious example $U = V \oplus W$, but are there any others?

Suppose we have a representation U as above. As a vector space, it can be (non-uniquely) identified with $V \oplus W$, so that for any $a \in A$ the corresponding operator $\rho_U(a)$ has block triangular form

$$\rho_U(a) = \begin{pmatrix} \rho_V(a) & f(a) \\ 0 & \rho_W(a) \end{pmatrix},$$

where $f : A \to \mathrm{Hom}_k(W, V)$.

(a) What is the necessary and sufficient condition on $f(a)$ under which $\rho_U(a)$ is a representation? Maps f satisfying this condition are called (1-)cocycles (of A with coefficients in $\mathrm{Hom}_k(W, V)$). They form a vector space denoted $Z^1(W, V)$.

(b) Let $X : W \to V$ be a linear map. The coboundary of X, dX, is defined to be the function $A \to \mathrm{Hom}_k(W, V)$ given by $dX(a) = \rho_V(a)X - X\rho_W(a)$. Show that dX is a cocycle, which vanishes if and only if X is a homomorphism of representations. Thus coboundaries form a subspace $B^1(W, V) \subset Z^1(W, V)$, which is isomorphic to $\mathrm{Hom}_k(W, V)/\mathrm{Hom}_A(W, V)$. The quotient $Z^1(W, V)/B^1(W, V)$ is denoted $\mathrm{Ext}^1(W, V)$.

(c) Show that if $f, f' \in Z^1(W, V)$ and $f - f' \in B^1(W, V)$ then the corresponding extensions U, U' are isomorphic representations of A. Conversely, if $\phi : U \to U'$ is an isomorphism which acts as the identity on V and projects onto Id_W, then $f - f' \in B^1(V, W)$. Such an isomorphism is called an **isomorphism of extensions***. Thus, the space $\mathrm{Ext}^1(W, V)$ "classifies" extensions of W by V.*

(d) Assume that W, V are finite-dimensional irreducible representations of A. For any $f \in \mathrm{Ext}^1(W, V)$, let U_f be the corresponding extension. Show that U_f is isomorphic to $U_{f'}$ as representations if and only if f and f' are proportional. Thus isomorphism classes (as representations) of nontrivial extensions of W by V (i.e., those not isomorphic to $W \oplus V$ as representations) are parametrized by the projective space $\mathbb{P}\mathrm{Ext}^1(W, V)$. In particular, every extension is trivial if and only if $\mathrm{Ext}^1(W, V) = 0$.

Problem 3.17. *(a) Let $A = \mathbb{C}[x_1, ..., x_n]$, and for $\mathbf{a}, = (a_1, \dots, a_n) \in \mathbb{C}^n$ let $V_{\mathbf{a}}$ be the one-dimensional representations in which x_i act by a_i. Find $\mathrm{Ext}^1(V_{\mathbf{a}}, V_{\mathbf{b}})$ and classify 2-dimensional representations of A.*

(b) Let B be the algebra over $\mathbb{C}$ generated by $x_1, ..., x_n$ with the defining relations $x_i x_j = 0$ for all i, j. Show that for $n > 1$ the algebra B has infinitely many non-isomorphic indecomposable representations.

Problem 3.18. *Let Q be a quiver without oriented cycles, and P_Q the path algebra of Q. Find irreducible representations of P_Q and compute* Ext^1 *between them. Classify 2-dimensional representations of P_Q.*

Problem 3.19. *Let A be an algebra, and V a representation of A. Let $\rho : A \to \mathrm{End}V$. A formal deformation of V is a formal series*

$$\tilde{\rho} = \rho_0 + t\rho_1 + ... + t^n\rho_n + ...,$$

where $\rho_i : A \to \mathrm{End}(V)$ are linear maps, $\rho_0 = \rho$, and $\tilde{\rho}(ab) = \tilde{\rho}(a)\tilde{\rho}(b)$.

If $b(t) = 1 + b_1 t + b_2 t^2 + ...$, where $b_i \in \mathrm{End}(V)$, and $\tilde{\rho}$ is a formal deformation of ρ, then $b\tilde{\rho}b^{-1}$ is also a deformation of ρ, which is said to be isomorphic to $\tilde{\rho}$.

(a) Show that if $\mathrm{Ext}^1(V,V) = 0$, then any deformation of ρ is trivial, i.e. isomorphic to ρ.

(b) Is the converse to (a) true? (consider the algebra of dual numbers $A = k[x]/x^2$).

4 Representations of finite groups: basic results

4.1 Maschke's Theorem

Theorem 4.1 (Maschke)**.** *Let G be a finite group and k an algebraically closed field whose characteristic does not divide $|G|$. Then the algebra $k[G]$ is semisimple. In particular, $k[G] = \oplus_i \mathrm{End}V_i$, where V_i are the irreducible representation of G.*

Proof. We need to show that if V is a finite-dimensional representation of G and $W \subset V$ is any subrepresentation, then there exists a subrepresentation $W' \subset V$ such that $V = W \oplus W'$ as the direct sum of representations.

Choose any complement $\hat{W}$ of W in V. (Thus $V = W \oplus \hat{W}$ as *vector spaces*, but not necessarily as *representations*.) Let P be the projection along $\hat{W}$ onto W, i.e., the operator on V defined by $P|_W = \mathrm{Id}$ and $P|_{\hat{W}} = 0$. Let

$$\overline{P} := \frac{1}{|G|}\sum_{g\in G} \rho(g)P\rho(g^{-1}),$$

where $\rho(g)$ is the action of g on V, and let

$$W' = \ker \overline{P}.$$

Now $\overline{P}|_W = \mathrm{Id}$ and $\overline{P}(V) \subseteq W$, so $\overline{P}^2 = \overline{P}$, so $\overline{P}$ is a projection along W'. Thus, $V = W \oplus W'$ as vector spaces.

Moreover, for any $h \in G$ and any $y \in W'$,

$$\overline{P}\rho(h)y = \frac{1}{|G|}\sum_{g\in G} \rho(g)P\rho(g^{-1}h)y = \frac{1}{|G|}\sum_{\ell\in G} \rho(h\ell)P\rho(\ell^{-1})y = \rho(h)\overline{P}y = 0,$$

so $\rho(h)y \in \ker \overline{P} = W'$. Thus, W' is invariant under the action of G and is therefore a subrepresentation of V. Thus, $V = W \oplus W'$ is the desired decomposition into subrepresentations. $\square$

Proposition 4.2. *Conversely, if $k[G]$ is semisimple, then the characteristic of k does not divide $|G|$.*

Proof. Write $k[G] = \bigoplus_{i=1}^r \mathrm{End}\, V_i$, where the V_i are irreducible representations and $V_1 = k$ is the trivial one-dimensional representation. Then

$$k[G] = k \oplus \bigoplus_{i=2}^{r} \mathrm{End}\, V_i = k \oplus \bigoplus_{i=2}^{r} d_i V_i,$$

where $d_i = \dim V_i$. By Schur's Lemma,

$$\mathrm{Hom}_{k[G]}(k, k[G]) = k\Lambda$$
$$\mathrm{Hom}_{k[G]}(k[G], k) = k\epsilon,$$

for nonzero homomorphisms $\epsilon : k[G] \to k$ and $\Lambda : k \to k[G]$ unique up to scaling. We can take ϵ such that $\epsilon(g) = 1$ for all $g \in G$, and Λ such that $\Lambda(1) = \sum_{g \in G} g$. Then

$$\epsilon \circ \Lambda(1) = \epsilon\left(\sum_{g \in G} g\right) = \sum_{g \in G} 1 = |G|.$$

If $|G| = 0$, then Λ has no left inverse, a contradiction. □

4.2 Characters

If V is a finite-dimensional representation of a finite group G, then its character is defined by the formula $\chi_V(g) = \mathrm{tr}|_V(\rho(g))$. Obviously, $\chi_V(g)$ is simply the restriction of the character $\chi_V(a)$ of V as a representation of the algebra $A = k[G]$ to the basis $G \subset A$, so it carries exactly the same information. The character is a *central* or *class function*: $\chi_V(g)$ depends only on the conjugacy class of g; i.e., $\chi_V(hgh^{-1}) = \chi_V(g)$.

Theorem 4.3. *If the characteristic of k does not divide $|G|$, characters of irreducible representations of G form a basis in the space $F_c(G, k)$ of class functions on G.*

Proof. By the Maschke theorem, $k[G]$ is semisimple, so by Theorem 3.12, the characters are linearly independent and are a basis of $(A/[A,A])^*$, where $A = k[G]$. It suffices to note that, as vector spaces over k,

$$(A/[A,A])^* \cong \{\varphi \in \mathrm{Hom}_k(k[G], k) \mid gh - hg \in \ker \varphi \ \forall g, h \in G\}$$
$$\cong \{f \in \mathrm{Fun}(G, k) \mid f(gh) = f(hg) \ \forall g, h \in G\},$$

which is precisely $F_c(G, k)$. □

Corollary 4.4. *The number of irreducible representations of G equals the number of conjugacy classes of G.*

Corollary 4.5. *Any representation of G is determined by its character; namely, $\chi_V = \chi_W$ implies $V \cong W$ if k has characteristic 0.*

4.3 Sum of squares formula

Theorem 4.6. *If G is a finite group and $\mathbb{C}[G]$ is its regular representation, then*

$$\mathbb{C}[G] \cong \bigoplus_{V \in \mathrm{Irrep}\, G} (\dim V) V,$$

where the direct sum is taken over all nonisomorphic irreducible representations.

Proof. By the Maschke theorem we can write $\mathbb{C}[G]$ as a direct sum of irreducible representations:

$$\mathbb{C}[G] \cong \bigoplus_{V \in \operatorname{Irrep} G} n_V V.$$

By the corollary to Schur's lemma, an endomorphism of representations on V must be a scalar times the identity. For any irreducible representation $V' \neq V$, any homomorphism of representations from V' to V must be 0. Thus,

$$\dim \operatorname{Hom}_G(V', V) = \begin{cases} 1 & \text{if } V' = V \\ 0 & \text{else.} \end{cases}$$

Comparing dimensions in

$$\operatorname{Hom}_G(\mathbb{C}[G], V) = \bigoplus_{V' \in \operatorname{Irrep} G} n_{V'} \operatorname{Hom}_G(V', V)$$

gives $\dim \operatorname{Hom}_G(\mathbb{C}[G], V) = n_V$. Furthermore,

$$\operatorname{Hom}_G(\mathbb{C}[G], V) = \{\varphi : G \to V \| \varphi(hg) = \rho_V(h)\varphi(g)\},$$

so such φ are of the form $\varphi(h) = \rho_V(h)v$ for an arbitrary vector $v \in V$. It follows that $\dim \operatorname{Hom}_G(\mathbb{C}[G], V) = \dim V$.

Thus, $n_V = \dim V$, and the theorem is proved. □

4.4 Examples

The following are examples of representations of finite groups over $\mathbb{C}$.

1. Finite abelian groups $G = \mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_k}$. Let $G^\vee$ be the set of irreducible representations of G. Every element of G forms a conjugacy class, so $|G^\vee| = |G|$. Recall that all irreducible representations over $\mathbb{C}$ (and algebraically closed fields in general) of commutative algebras and groups are one-dimensional. Thus, $G^\vee$ is an abelian group: if $\rho_1, \rho_2 : G \to \mathbb{C}^*$ are irreducible representations then so are $\rho_1(g)\rho_2(g)$ and $\rho_1(g)^{-1}$. $G^\vee$ is called the *dual* or *character group* of G.

   For given $n \geq 2$, define $\rho : \mathbb{Z}_n \to \mathbb{C}^*$ by $\rho(m) = e^{2\pi i m/n}$. Then $\mathbb{Z}_n^\vee = \{\rho^k : k = 0, \ldots, n-1\}$, so $\mathbb{Z}_n^\vee \cong \mathbb{Z}_n$. In general,

   $$(G_1 \times G_2 \times \cdots \times G_n)^\vee = G_1^\vee \times G_2^\vee \times \cdots \times G_n^\vee,$$

   so $G^\vee \cong G$ for any finite abelian group G. This isomorphism is, however, noncanonical: the particular decomposition of G as $\mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_k}$ is not unique as far as which elements of G correspond to $\mathbb{Z}_{n_1}$, etc. is concerned. On the other hand, $G \cong (G^\vee)^\vee$ is a canonical isomorphism, given by $\varphi : G \to (G^\vee)^\vee$, where $\varphi(g)(\chi) = \chi(g)$.

2. The symmetric group S_3. In S_n, conjugacy classes are based on cycle decomposition sizes: two permutations are conjugate iff they have the same number of cycles of each length. For S_3, there are 3 conjugacy classes, so there are 3 different irreducible representations over $\mathbb{C}$. If their dimensions are d_1, d_2, d_3, then $d_1^2 + d_2^2 + d_3^2 = 6$, so S_3 must have two 1-dimensional and one 2-dimensional representations. The 1-dimensional representations are the trivial representation $\rho(\sigma) = 1$ and the sign representation $\rho(\sigma) = (-1)^\sigma$.

The 2-dimensional representation can be visualized as representing the symmetries of the equilateral triangle with vertices 1, 2, 3 at the points $(\cos 120°, \sin 120°)$, $(\cos 240°, \sin 240°)$, $(1, 0)$ of the coordinate plane, respectively. Thus, for example,

$$\rho((12)) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \rho((123)) = \begin{pmatrix} \cos 120° & -\sin 120° \\ \sin 120° & \cos 120° \end{pmatrix}.$$

To show that this representation is irreducible, consider any subrepresentation V. V must be the span of a subset of the eigenvectors of $\rho((12))$, which are the nonzero multiples of $(1, 0)$ and $(0, 1)$. V must also be the span of a subset of the eigenvectors of $\rho((123))$, which are the nonzero multiples of $(1, i)$ and $(i, 1)$. Thus, V must be either $\mathbb{C}^2$ or 0.

3. The quaternion group $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$, with defining relations

$$i = jk = -kj, \qquad j = ki = -ik, \qquad k = ij = -ji, \qquad -1 = i^2 = j^2 = k^2.$$

The 5 conjugacy classes are $\{1\}, \{-1\}, \{\pm i\}, \{\pm j\}, \{\pm k\}$, so there are 5 different irreducible representations, the sum of the squares of whose dimensions is 8, so their dimensions must be 1, 1, 1, 1, and 2.

The center $Z(Q_8)$ is $\{\pm 1\}$, and $Q_8/Z(Q_8) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. The four 1-dimensional irreducible representations of $\mathbb{Z}_2 \times \mathbb{Z}_2$ can be "pulled back" to Q_8. That is, if $q : Q_8 \to Q_8/Z(Q_8)$ is the quotient map, and ρ any representation of $Q_8/Z(Q_8)$, then $\rho \circ q$ gives a representation of Q_8.

The 2-dimensional representation is $V = \mathbb{C}^2$, given by $\rho(-1) = -\mathrm{Id}$ and

$$\rho(i) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \rho(j) = \begin{pmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{pmatrix}, \qquad \rho(k) = \begin{pmatrix} 0 & -\sqrt{-1} \\ -\sqrt{-1} & 0 \end{pmatrix}.$$

These are the Pauli matrices, which arise in quantum mechanics.

4. The symmetric group S_4. The order is 24, and there are 5 conjugacy classes: $e, (12), (123), (1234), (12)(34)$. Thus the sum of the squares of the dimensions of 5 irreducible representations is 24. As with S_3, there are two of dimension 1: the trivial and sign representations, $\mathbb{C}$ and $\mathbb{C}_{-1}$. The other three must have dimensions 2, 2, and 3. Because $S_3 \cong S_4/V$, where V is the Viergruppe $\{e, (12)(34), (13)(24), (14)(23)\}$, the 2-dimensional representation of S_3 can be pulled back to the 2-dimensional representation of S_4, which we will call $\mathbb{C}^2$.

We can consider S_4 as the group of rotations of a cube (acting by permuting the interior diagonals); this gives the 3-dimensional representation $\mathbb{C}^3_+$.

The last 3-dimensional representation is $\mathbb{C}^3_-$, the product of $\mathbb{C}^3_+$ with the sign representation, or equivalently the permutation group of a regular tetrahedron. $\mathbb{C}^3_+$ and $\mathbb{C}^3_-$ are different, for if g is a transposition, $\det g|_{\mathbb{C}^3_+} = 1$ while $\det g|_{\mathbb{C}^3_-} = (-1)^3 = -1$.

4.5 Duals and tensor products of representations

If V is a representation of a finite group G, then V^* is also a representation, via

$$\rho_{V^*}(g) = (\rho_V(g)^*)^{-1} = (\rho_V(g)^{-1})^* = \rho_V(g^{-1})^*.$$

The character is $\chi_{V^*}(g) = \chi_V(g^{-1})$.

For complex representations, $\chi_V(g) = \sum \lambda_i$, where the λ_i are the eigenvalues of g in V. These eigenvalues must be roots of unity because $\rho(g)^{|G|} = \rho(g^{|G|}) = \rho(e) = \mathrm{Id}$. Thus

$$\chi_{V^*}(g) = \chi_V(g^{-1}) = \sum \lambda_i^{-1} = \sum \overline{\lambda_i} = \overline{\sum \lambda_i} = \overline{\chi_V(g)}.$$

In particular, $V \cong V^*$ as *representations* (not just as vector spaces) iff $\chi_V(g) \in \mathbb{R}$ for all $g \in G$.

If V, W are representations of G, then $V \otimes W$ is also a representation, via

$$\rho_{V\otimes W}(g) = \rho_V(g) \otimes \rho_W(g).$$

It is an interesting problem to decompose $V \otimes W$ into the direct sum of irreducible representations.

4.6 Orthogonality of characters

We define the Hermitian inner product form on $\mathrm{F}_c(G, \mathbb{C})$ (the space of central functions) by

$$(f_1, f_2) = \frac{1}{|G|} \sum_{g \in G} f_1(g)\overline{f_2(g)}.$$

The following theorem says that characters of irreducible representations of G form an orthonormal basis of $F_c(G, \mathbb{C})$ under this inner product.

Theorem 4.7. *For any representations* V, W

$$(\chi_V, \chi_W) = \dim Hom(V, W),$$

and

$$(\chi_V, \chi_W) = \begin{cases} 1, \text{ if } V \cong W, \\ 0, \text{ if } V \ncong W \end{cases}$$

if V, W *are irreducible.*

Proof. By the definition

$$\begin{aligned}(\chi_V, \chi_W) &= \frac{1}{|G|} \sum_{g \in G} \chi_V(g)\overline{\chi_W(g)} = \frac{1}{|G|} \sum_{g \in G} \chi_V(g)\chi_{W^*}(g) \\ &= \frac{1}{|G|} \sum_{g \in G} \chi_{V\otimes W^*}(g) = \mathrm{tr}\,|_{V\otimes W^*}(P),\end{aligned}$$

where $P = \frac{1}{|G|} \sum_{g \in G} g \in Z(\mathbb{C}[G])$. (Here $Z(\mathbb{C}[G])$ denotes the center of $\mathbb{C}[G]$). If X is an irreducible representation of G then

$$P|_X = \begin{cases} \mathrm{Id}, \text{ if } X = \mathbb{C}, \\ 0, \quad X \neq \mathbb{C}. \end{cases}$$

Therefore, for any representation X the operator $P|_X$ is the G-invariant projector onto the subspace X^G of G-invariants in X. Thus,

$$\begin{aligned}\mathrm{tr}\,|_{V\otimes W^*}(P) &= \dim \mathrm{Hom}_G(\mathbb{C}, V \otimes W^*) \\ &= \dim(V \otimes W^*)^G = \dim \mathrm{Hom}_G(W, V).\end{aligned}$$

□

Here is another "orthogonality formula" for characters, in which summation is taken over irreducible representations rather than group elements.

Theorem 4.8. *Let $g, h \in G$, and let Z_g denote the centralizer of g in G. Then*

$$\sum_V \chi_V(g)\overline{\chi_V(h)} = \begin{array}{l} |Z_g| if g is conjugate to h \\ 0, otherwise. \end{array}$$

where the summation is taken over all irreducible representations of G.

Proof. As noted above, $\overline{\chi_V(h)} = \chi_{V^*}(h)$, so the left hand side equals (using Maschke's theorem):

$$\sum_V \chi_V(g)\chi_{V^*}(h) = \mathrm{tr}|_{\mathbb{C}[G]}(x \mapsto gxh^{-1}).$$

If g and h are not conjugate, this trace is clearly zero, since the matrix of the operator $x \to gxh^{-1}$ in the basis of group elements has zero diagonal entries. On the other hand, if g and h are in the same conjugacy class, the trace is equal to the number of elements x such that $x = gxh^{-1}$, i.e., the order of the centralizer Z_g of g. We are done. □

4.7 Unitary representations. Another proof of Maschke's theorem for complex representations

Definition 4.9. A unitary finite dimensional representation of a group G is a representation of G on a complex finite dimensional vector space V over $\mathbb{C}$ equipped with a G-invariant positive definite Hermitian form $(,)$, i.e. such that $\rho_V(g)$ are unitary operators: $(\rho_V(g)v, \rho_V(g)w) = (v, w)$.

Remark 4.10. Not any finite dimensional representation admits a unitary structure.

Theorem 4.11. *If G is finite, then any finite dimensional representation of G has a unitary structure. If the representation if irreducible, this structure is unique up to scaling.*

Proof. Take any positively defined form B on V and define another form $\overline{B}$ as follows:

$$\overline{B}(v, w) = \sum_{g \in G} B(gv, gw)$$

Then $\overline{B}$ is a positive definite Hermitian form on V, and $\rho_V(g)$ are unitary operators. If V is an irreducible representation and B_1, B_2 are two Hermitian forms on V, then $B_1(v, w) = B_2(v, Aw)$ for some homomorphism $A : V \to V$. This implies $A = \lambda \mathrm{Id}$, $\lambda \in \mathbb{R}$. □

Theorem 4.12. *A finite dimensional unitary representation V of any group G is completely reducible.*

Proof. Let W be a subrepresentation of V. Let $W^\perp$ be the orthogonal complement of W in V under the Hermitian inner product. Then $W^\perp$ is a subrepresentation of W, and $V = W \oplus W^\perp$. This implies that V is completely reducible. □

4.8 Orthogonality of matrix elements

Let V be an irreducible representation of G, and $v_1, v_2, \ldots, v_n$ be an orthonormal basis of V under the Hermitian form. The matrix elements of V are $T^V_{ij}(x) = (v_i, \rho_V(x)v_j)$.

Proposition 4.13. *(1) Matrix elements of nonisomorphic representations are orthogonal in $F(G, \mathbb{C})$ under the form $(f, g) = \frac{1}{|G|}\sum_{x \in G} f(x)\overline{g(x)}$.*

(2) $(t^V_{ij}, t^V_{i'j'}) = \delta_{ii'}\delta_{jj'} \cdot \frac{1}{\dim V}$

Proof. Let V and W be two representations of G. Take $\{v_i\}$ to be an orthonormal basis of V and $\{w_i\}$ to be an orthonormal basis of W. Putting $P = \sum_{x\in G} x$, we have

$$\sum_{x\in G}\langle v_i, xv_j\rangle\overline{\langle w_{i'}, xw_{j'}\rangle} = \sum_{x\in G}\langle v_i, xv_j\rangle\langle w^*_{i'}, xw^*_{j'}\rangle = \langle v_i \otimes w^*_{i'}, P(v_j \otimes w^*_{j'})\rangle$$

□

If $V \neq W$, this is zero, since P projects to the trivial representation. If $V = W$, we need to consider $\langle v_i \otimes v^*_{i'}, P(v_j \otimes v^*_{j'})\rangle$. We have

$$\begin{aligned} V \otimes V^* &= \mathbb{C} \oplus L \\ \mathbb{C} &= \mathrm{span}(\sum v_k \otimes v^*_k) \\ L &= \mathrm{span}(\{\sum_{\sum a_{kl}=0} a_{kl} v_k \otimes v^*_l\}) \end{aligned}$$

The projection of $v_i \otimes v^*_{i'}$ to $\mathbb{C} \subset \mathbb{C} \oplus L$ is

$$\frac{1}{\dim V}\sum v_k \otimes v^*_k \delta_{ii'}$$

This shows that

$$\langle v_i \otimes v_{i'}, P(v_j \otimes v^*_{j'}))\rangle = \frac{\delta_{ii'}\delta_{jj'}}{\dim V}$$

which finishes the proof.

4.9 Character tables, examples

The characters of all the irreducible representations of a finite group can be arranged into a character table, with conjugacy classes of elements as the columns, and characters as the rows. More specifically, the first row in a character table lists representatives of conjugacy classes, the second one the numbers of elements in the conjugacy classes, and the other rows are the values of the characters on the conjugacy classes. Due to theorems 4.7 and 4.8 the rows and columns of a character table are orthonormal with respect to the appropriate inner products.

Note that in any character table, the row corresponding to the trivial representation consists of ones, and the column corresponding to the neutral element consists of the dimensions of the representations.

Here is, for example, the character table of S_3 :

S_3	Id	(12)	(123)
#	1	3	2
$\mathbb{C}$	1	1	1
$\mathbb{C}_-$	1	-1	1
$\mathbb{C}^2$	2	0	1

It is obtained by explicitly computing traces in the irreducible representations.

For another example consider A_4, the group of even permutations of 4 items. There are three one-dimensional representations. Since there are four conjugacy classes in total, there is one more irreducible representation of dimension 3. Finally, the character table is

A_4	Id	(123)	(132)	(12)(34)
#	1	4	4	3
$\mathbb{C}$	1	1	1	1
$\mathbb{C}_\epsilon$	1	ϵ	ϵ^2	1
$\mathbb{C}_{\epsilon^2}$	1	ϵ^2	ϵ	1
$\mathbb{C}^3$	3	0	0	-1

where $\epsilon = \exp(\frac{2\pi i}{3})$.

The last row can be computed using the orthogonality of rows. Another way to compute the last row is to note that $\mathbb{C}^3$ is the representation of A_4 by rotations of the regular tetrahedron: in this case $(123), (132)$ are the rotations by 120^0 and 240^0 around a perpendicular to a face of the tetrahedron, while $(12)(34)$ is the rotation by 180^0 around an axis perpendicular to two opposite edges.

Example 4.14. The following three character tables are of Q_8, S_4, and A_5 respectively.

Q_8	1	-1	i	j	k
#	1	1	2	2	2
$\mathbb{C}_{++}$	1	1	1	1	1
$\mathbb{C}_{+-}$	1	1	1	-1	-1
$\mathbb{C}_{-+}$	1	1	-1	1	-1
$\mathbb{C}_{--}$	1	1	-1	-1	1
$\mathbb{C}^2$	2	2	0	0	0

S_4	Id	(12)	(12)(34)	(123)	(1234)
#	1	6	3	8	6
$\mathbb{C}$	1	1	1	1	1
$\mathbb{C}_-$	1	-1	1	1	-1
$\mathbb{C}^2$	2	0	2	-1	0
$\mathbb{C}^3_+$	3	-1	-1	0	1
$\mathbb{C}^3_-$	3	1	-1	0	-1

A_5	Id	(123)	(12)(34)	(12345)	(13245)
#	1	20	15	12	12
$\mathbb{C}$	1	1	1	1	1
$V_3^+ = \mathbb{C}^3_{(1)}$	3	0	-1	$\frac{1+\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$
$V_3^- = \mathbb{C}^3_{(2)}$	3	0	-1	$\frac{1-\sqrt{5}}{2}$	$\frac{1+\sqrt{5}}{2}$
$V_4 = \mathbb{C}^4$	4	1	0	-1	-1
$V_5 = \mathbb{C}^5$	5	-1	1	0	0

Indeed, the computation of the characters of the 1-dimensional representations is straightforward.

The character of the 2-dimensional representation of Q_8 is obtained by using character orthogonality.

For S_4, the 2-dimensional irreducible representation is obtained from the 2-dimensional irreducible representation of S_3 via the surjective homomorphism $S_4 \to S_3$, which allows to obtain its character from the character table of S_3.

The character of the 3-dimensional representation $\mathbb{C}^3_+$ is computed from its geometric realization by rotations of the cube. Namely, by rotating the cube, S_4 permuts the main diagonals. Thus (12) is rotaion by 180^0 around an axis that is perpendicular to two opposite edges, (12)(34) is rotation by 180^0 around an axis that is perpendicular to two opposite faces, (123) is rotation around a main diagonal by 120^0, and (1234) is rotation by 90^0 around an axis that is perpendicular to two opposite faces; this allows us to compute the traces easily, using the fact that the trace of a rottation by the angle ϕ in $\mathbb{R}^3$ is $1 + 2\cos\phi$. Now the character of $\mathbb{C}^3_-$ is found by multiplying the character of $\mathbb{C}^3_+$ by the character of the sign representation.

Finally, we explain how to obtain the character table of A_5. The group A_5 is the group of rotations of the regular icosahedron. Thus it has a 3-dimensional "rotation representation" V_3^+, in which (12)(34) is rotation by 180^0 around an axis perpendicular to two opposite edges, (123) is rotation by 120^0 around an axis perpendicular to two opposite faces, and (12345), (13254) are rotations by 72^0, respectively 144^0, around axes going through two opposite vertices. The character of this representation is computed from this description in a straightforward way.

Another representation of A_5, which is also 3-dimensional, is V_3^+ twisted by the automorphism of A_5 given by coinjugation by (12) inside S_5. This representation is denoted by V_3^-. It has the same character as V_3^+, except that the conjugacy classes (12345) and (13245) are interchanged.

There are two remaining irreducible representations, and by the sum of squares formula their dimensions are 4 and 5. So we call them V_4 and V_5.

The representation V_4 is realized on the space of functions on the set $\{1, 2, 3, 4, 5\}$ with zero sum of values (where A_5 acts by permutations). The character of this representation is equal to the character of the 5-dimensional permutation representation minus the character of the 1-dimensional trivial representation (constant functions). The former at an element g equals to the number of items among 1,2,3,4,5 which are fixed by g.

The representation V_5 is realized on the space of functions on pairs of opposite vertices of the icosahedron which has zero sum of values. The character of this representation is computed similarly to the character of V_4, or from the orthogonality formula.

4.10 Computing tensor product multiplicities and restriction multiplicities using character tables

Character tables allow us to compute the tensor product multiplicities N_{ij}^k using

$$V_i \otimes V_j = \sum N_{ij}^k V_k, \quad N_{ij}^k = (\chi_i \chi_j, \chi_k)$$

Example 4.15. The following tables represent computed tensor product multiplicities of irreducible representations of S_3, S_4, and A_4 respectively.

	$\mathbb{C}$	$\mathbb{C}_-$	$\mathbb{C}^2$
$\mathbb{C}$	$\mathbb{C}$	$\mathbb{C}_-$	$\mathbb{C}^2$
$\mathbb{C}_-$	$\mathbb{C}_-$	$\mathbb{C}$	$\mathbb{C}^2$
$\mathbb{C}^2$	$\mathbb{C}^2$	$\mathbb{C}^2$	$\mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}^2$

	$\mathbb{C}_+$	$\mathbb{C}_-$	$\mathbb{C}^2$	$\mathbb{C}^3_+$	$\mathbb{C}^3_-$
$\mathbb{C}_+$	$\mathbb{C}_+$	$\mathbb{C}_-$	$\mathbb{C}^2$	$\mathbb{C}^3_+$	$\mathbb{C}^3_-$
$\mathbb{C}_-$	$\mathbb{C}_-$	$\mathbb{C}_+$	$\mathbb{C}^2$	$\mathbb{C}^3_-$	$\mathbb{C}^3_+$
$\mathbb{C}^2$	$\mathbb{C}^2$	$\mathbb{C}^2$	$\mathbb{C} \oplus \mathbb{C}_- \oplus \mathbb{C}^3_-$	$\mathbb{C}^3_+ \oplus \mathbb{C}^3_-$	$\mathbb{C}^3_+ \oplus \mathbb{C}^3_-$
$\mathbb{C}^3_+$	$\mathbb{C}^3_+$	$\mathbb{C}^3_-$	$\mathbb{C}^3_+ \oplus \mathbb{C}^3_-$	$\mathbb{C}^2 \oplus \mathbb{C}_+ \oplus \mathbb{C}^3_+ \oplus \mathbb{C}^3_-$	$\mathbb{C}^2 \oplus \mathbb{C}_- \oplus \mathbb{C}^3_+ \oplus \mathbb{C}^3_-$
$\mathbb{C}^3_-$	$\mathbb{C}^3_-$	$\mathbb{C}^3_+$	$\mathbb{C}^3_+ \oplus \mathbb{C}^3_-$	$\mathbb{C}^2 \oplus \mathbb{C}_- \oplus \mathbb{C}^3_+ \oplus \mathbb{C}^3_-$	$\mathbb{C}^2 \oplus \mathbb{C}_+ \oplus \mathbb{C}^3_+ \oplus \mathbb{C}^3_-$

	$\mathbb{C}$	V_3^+	V_3^-	V_4	V_5
$\mathbb{C}$	$\mathbb{C}$	V_3^+	V_3^-	V_4	V_5
V_3^+		$\mathbb{C} \oplus V_5 \oplus V_3^+$	$V_4 \oplus V_5$	$V_3^- \oplus V_4 \oplus V_5$	$V_3^+ \oplus V_3^- \oplus V_4 \oplus V_5$
V_3^-			$\mathbb{C} \oplus V_5 \oplus V_3^+$	$V_3^+ \oplus V_4 \oplus V_5$	$V_3^+ \oplus V_3^- \oplus V_4 \oplus V_5$
V_4				$V_3^+ \oplus V_3^- \oplus \mathbb{C} \oplus V_4 \oplus V_5$	$V_3^+ \oplus V_3^- \oplus 2V_5 \oplus V_4$
V_5					$\mathbb{C} \oplus V_3^+ \oplus V_3^- \oplus 2V_4 \oplus 2V_5$

4.11 Problems

Problem 4.16. *Let G be the group of symmetries of a regular N-gon (it has $2N$ elements).*

(a) Describe all irreducible complex representations of this group.

(b) Let V be the 2-dimensional complex representation of G obtained by complexification of the standard representation on the real plane (the plane of the polygon). Find the decomposition of $V \otimes V$ in a direct sum of irreducible representations.

Problem 4.17. *Let G be the group of 3 by 3 matrices over F_p which are upper triangular and have 1-s on the diagonal, under multiplication (its order is of course p^3). It is called the Heisenberg group. For any complex number z such that $z^p = 1$ we define a representation of G on the space V of complex functions on F_p, by*

$$(\rho \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} f)(x) = f(x-1),$$

$$(\rho \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} f)(x) = z^x f(x).$$

(note that z^x makes sense since $z^p = 1$).

(a) Show that such a representation exists and is unique, and compute $\rho(g)$ for all $g \in G$.

(b) Denote this representation by R_z. Show that R_z is irreducible if and only if $z \neq 1$.

(c) Classify all 1-dimensional representations of G. Show that R_1 decomposes into a direct sum of 1-dimensional representations, where each of them occurs exactly once.

(d) Use (a)-(c) and the "sum of squares" formula to classify all irreducible representations of G.

Problem 4.18. *Let V be a finite dimensional complex vector space, and $GL(V)$ be the group of invertible linear transformations of V. Then $S^n V$ and $\Lambda^m V$ ($m \leq dim(V)$) are representations of $GL(V)$ in a natural way. Show that they are irreducible representations.*

Hint: Choose a basis $\{e_i\}$ in V. Find a diagonal element H of $GL(V)$ such that $\rho(H)$ has distinct eigenvalues. (where ρ is one of the above representations). This shows that if W is a subrepresentation, then it is spanned by a subset S of a basis of eigenvectors of $\rho(H)$. Use the invariance of W under the operators $\rho(1 + E_{ij})$ (where E_{ij} is defined by $E_{ij} e_k = \delta_{jk} e_i$) for all $i \neq j$ to show that if the subset S is nonempty, it is necessarily the entire basis.

Problem 4.19. *Recall that the adjacency matrix of a graph Γ (without multiple edges) is the matrix in which the ij-th entry is 1 if the vertices i and j are connected with an edge, and zero otherwise. Let Γ be a finite graph whose automorphism group is nonabelian. Show that the adjacency matrix of Γ must have repeated eigenvalues.*

Problem 4.20. *Let I be the set of vertices of a regular icosahedron ($|I| = 12$). Let $F(I)$ be the space of complex functions on I. Recall that the group $G = A_5$ of even permutations of 5 items acts on the icosahedron, so we have a 12-dimensional representation of G on $F(I)$.*

(a) Decompose this representation in a direct sum of irreducible representations (i.e., find the multiplicities of occurrence of all irreducible representations).

(b) Do the same for the representation of G on the space of functions on the set of faces and the set of edges of the icosahedron.

Problem 4.21. *Let F be a finite field with q elements, and G be the group of inhomogeneous linear transformations, $x \to ax + b$, over F (i.e., $a \in F^\times, b \in F$). Find all irreducible complex representations of G, and compute their characters. Compute the tensor products of irreducible representations.*

Hint. Let V be the representation of G on the space of functions on F with sum of all values equal to zero. Show that V is an irreducible representation of G.

Problem 4.22. *Let $G = SU(2)$ (unitary 2 by 2 matrices with determinant 1), and $V = \mathbb{C}^2$ the standard 2-dimensional representation of $SU(2)$. We consider V as a real representation, so it is 4-dimensional.*

(a) Show that V is irreducible (as a real representation).

(b) Let H be the subspace of $\mathrm{End}_{\mathbb{R}}(V)$ consisting of endomorphisms of V as a real representation. Show that H is 4-dimensional and closed under multiplication. Show that every nonzero element in H is invertible, i.e. H is an algebra with division.

(c) Find a basis $1, i, j, k$ of H such that 1 is the unit and $i^2 = j^2 = k^2 = -1$, $ij = -ji = k, jk = -kj = i, ki = -ik = j$. Thus we have that Q_8 is a subgroup of the group $H^\times$ of invertible elements of H under multiplication.

The algebra H is called the quaternion algebra.

(d) For $q = a+bi+cj+dk$, $a, b, c, d \in \mathbb{R}$, let $\bar{q} = a-bi-cj-dk$, and $||q||^2 = q\bar{q} = a^2+b^2+c^2+d^2$. Show that $\overline{q_1 q_2} = \bar{q}_2\bar{q}_1$, and $||q_1q_2|| = ||q_1|| \cdot ||q_2||$.

(e) Let G be the group of quaternions of norm 1. Show that this group is isomorphic to $SU(2)$. (Thus $SU(2)$ is the 3-dimensional sphere).

(f) Consider the action of G on the space $V \subset H$ spanned by i, j, k, by $x \to qxq^{-1}$, $q \in G$, $x \in V$. Since this action preserves the norm on V, we have a homomorphism $h : SU(2) \to SO(3)$. Show that this homomorphism is surjective and that its kernel is $\{1, -1\}$.

Problem 4.23. *Using the classification of finite subgroups of $SO(3)$ (M. Artin, "Algebra", p.184), classify finite subgroups of $SU(2)$ (use the homomorphism $SU(2) \to SO(3)$).*

Problem 4.24. *Find the characters and tensor products of irreducible complex representations of the Heisenberg group from Problem 4.17.*

Problem 4.25. *Let G be a finite group, and V a complex representation of G which is faithful, i.e. the corresponding map $G \to GL(V)$ is injective. Show that any irreducible representation of G occurs inside S^nV (and hence inside $V^{\otimes n}$) for some n.*

Problem 4.26. *This problem is about an application of representation theory to physics (elasticity theory). We first describe the physical motivation and then state the mathematical problem.*

Imagine a material which occupies a certain region U in the physical space $V = \mathbb{R}^3$ (a space with a positive definite inner product). Suppose the material is deformed. This means, we have *applied a diffeomorphism (=change of coordinates) $g : U \to U'$. The question in elasticity theory is how much stress in the material this deformation will cause.*

For every point P, let $A_P : V \to V$ be defined by $A_P = dg(P)$. A_P is nondegenerate, so it has a polar decomposition $A_P = D_P O_P$, where O_P is orthogonal and D_P is symmetric. The matrix O_P characterizes the rotation part of A_P (which clearly produces no stress), and D_P is the distortion part, which actually causes stress. If the deformation is small, D_P is close to 1, so $D_P = 1 + d_P$, where d_P is a small symmetric matrix, i.e. an element of S^2V. This matrix is called the deformation tensor at P.

Now we define a stress tensor, which characterizes stress. Let v be a small nonzero vector in V, and σ a small disk perpendicular to v centered at P of area $||v||$. Let F_v be the force with which the part of the material on the v-side of σ acts on the part on the opposite side. It is easy to deduce from Newton's laws that F_v is linear in v, so there exists a linear operator $S_P : V \to V$ such that $F_v = S_P v$. It is called the stress tensor.

An elasticity law is an equation $S_P = f(d_P)$, where f is a function. The simplest such law is a linear law (Hooke's law): $f : S^2V \to End(V)$ is a linear function. In general, such a function is defined by $9 \cdot 6 = 54$ parameters, but we will show there are actually only two essential ones – the compression modulus K and the shearing modulus μ. For this purpose we will use representation theory.

Recall that the group $SO(3)$ of rotations acts on V, so S^2V, $End(V)$ are representations of this group. The laws of physics must be invariant under this group (Galileo transformations), so f must be a homomorphism of representations.

(a) Show that $End(V)$ admits a decomposition $\mathbb{R} \oplus V \oplus W$, where $\mathbb{R}$ is the trivial representation, V is the standard 3-dimensional representation, and W is a 5-dimensional representation of SO(3). Show that $S^2V = \mathbb{R} \oplus W$

(b) Show that V and W are irreducible, even after complexification. Deduce using Schur's lemma that S_P is always symmetric, and for $x \in \mathbb{R}, y \in W$ one has $f(x + y) = Kx + \mu y$ for some real numbers K, μ.

In fact, it is clear from physics that K, μ are positive.

5 Representations of finite groups: further results

5.1 Frobenius-Schur indicator

Suppose that G is a finite group and V is an irreducible representation of G over $\mathbb{C}$. We call V

- complex, if $V \ncong V^*$,
- real, if V has a nondegenerate symmetric form invariant under G,
- quaternionic, if V has a nondegenerate skew form invariant under G.

If we consider $\mathrm{End}_{\mathbb{R}} V$, then it is $\mathbb{C}$ for complex V, $\mathrm{Mat}_2(\mathbb{R})$ for real V, and $\mathbb{H}$ for quaternionic V, which suggests the names above.

Example 5.1. For S_3 there all three irreducible representations $\mathbb{C}_+, \mathbb{C}_-, \mathbb{C}^2$ are real. For S_4 there are five irreducible representations $\mathbb{C}_+$, $\mathbb{C}_-$, $\mathbb{C}^2$, $\mathbb{C}^3_+$, $\mathbb{C}^3_-$, which are all real. Similarly, all five irreducible representations of A_5 – $\mathbb{C}$, V_3^+, V_3^-, V_4, V_5 are real. As for Q_8, its one-dimensional representations are real, and the two-dimensional one is quaternionic.

Theorem 5.2. *(Frobenius-Schur) The number of involutions (=elements of order 2) in G is equal to the sum of dimensions of real representations minus the sum of dimensions of quaternionic representations.*

Proof. Let $A : V \to V$ have eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. We have

$$\begin{aligned}\mathrm{Tr}|_{S^2V}(A \otimes A) &= \sum_{i \leq j} \lambda_i \lambda_j \\ \mathrm{Tr}|_{\Lambda^2V}(A \otimes A) &= \sum_{i < j} \lambda_i \lambda_j\end{aligned}$$

Thus,

$$\mathrm{Tr}|_{S^2V}(A \otimes A) - \mathrm{Tr}|_{\Lambda^2V}(A \otimes A) = \sum_{1 \leq i \leq n} \lambda_i^2 = \mathrm{Tr}(A^2).$$

Thus for $g \in G$ we have

$$\chi_V(g^2) = \chi_{S^2V}(g) - \chi_{\Lambda^2V}(g)$$

Therefore,

$$\chi_V(\sum_{g \in G} g^2) = |G| \begin{cases} 1, & \text{if } V \text{ is real} \\ -1, & \text{if } V \text{ is quaternionic} \\ 0, & \text{if } V \text{ is complex} \end{cases}$$

Finally, the number of involutions in G equals

$$\frac{1}{|G|} \sum_V \dim V \chi_V(\sum_{g \in G} g^2) = \sum_{\text{real } V} \dim V - \sum_{\text{quat } V} \dim V$$

□

5.2 Frobenius determinant

Enumerate the elements of a finite group G as follows: $g_1, g_2, \ldots, g_n$. Introduce n variables indexed with the elements of G :

$$x_{g_1}, x_{g_2}, \ldots, x_{g_n}.$$

Definition 5.3. Consider the matrix X_G with entries $a_{ij} = x_{g_i g_j}$. The determinant of X_G is some polynomial of degree n of $x_{g_1}, x_{g_2}, \ldots, x_{g_n}$ that is called Frobenius determinant.

The following theorem, discovered by Dedekind and proved by Frobenius, became the starting point for creation of representation theory.

Theorem 5.4.

$$\det X_G = \prod_{j=1}^{r} P_j(\mathbf{x})^{\deg P_j}$$

for some pairwise non-proportional irreducible polynomials $P_j(\mathbf{x})$.

We will need the following rather simple lemma.

Lemma 5.5. *Let Y be a $n \times n$ matrix with entries y_{ij}. Then $\det Y$ is an irreducible polynomial of $\{y_{ij}\}$.*

Proof. Suppose that $\det Y = q_1 q_2 \dots q_k$, $k \geq 2$ where q_i are irreducible homogeneous polynomials. Because $\det Y$ is linear on each column, there is exactly one q_i that depends on any column.

Thus q_1 is a linear function on some column, say j^{th}. Pick $\{y_{ij}\}_{i=1}^n$ so that $q_1 = 0$. Then any matrix Y with such j^{th} column must have $\det Y = 0$. This is clearly false for $n \geq 2$. Contradiction. □

Now we are ready to proceed to the proof Theorem 5.4.

Proof. Let $V = \mathbb{C}[G]$ be the regular representation of G. Consider operator-valued polynomial

$$L(\mathbf{x}) = \sum_{g \in G} x_g \rho(g),$$

where $\rho(g) \in \text{End} V$ is induced by g. The action of $L(\mathbf{x})$ on element $h \in G$ is

$$L(\mathbf{x})h = \sum_{g \in G} x_g \rho(g) h = \sum_{g \in G} x_g g h = \sum_{z \in G} x_{zh^{-1}} z$$

So the matrix of the linear operator $L(\mathbf{x})$ in the basis $g_1, g_2, \dots, g_n$ which is X_G with permuted columns and, hence, the same determinant up to sign. Further, we have

$$\det_V L(\mathbf{x}) = \prod_{i=1}^{r} (\det_{V_i} L(\mathbf{x}))^{\dim V_i}.$$

We set $P_i = \det_{V_i} L(\mathbf{x})$. Recall that $\mathbb{C}[G] = \bigoplus_{i=1}^r \text{End}\, V_i$. Let $\{e_{im}\}$ be bases of V_i and $E_{i,jk} \in \text{End}\, V_i$ be the matrix units in these bases. Then $\{E_{i,jk}\}$ is a basis of $\mathbb{C}[G]$ and

$$L(\mathbf{x})|_{V_i} = \sum y_{i,jk} E_{i,jk},$$

where $y_{i,jk}$ are new coordinates on $\mathbb{C}[G]$ related to x_g by a linear transformation. Then

$$P_i(\mathbf{x}) = \det|_{V_i} L(\mathbf{x}) = \det(y_{i,jk})$$

Hence, P_i are irreducible (by lemma 5.5) and not proportional. □

5.3 Algebraic integers

We are now passing to deeper results in representation theory of finite groups. These results require the theory of algebraic numbers, which we will now briefly review.

Definition 5.6. $z \in \mathbb{C}$ is an **algebraic integer** if z is a root of a monic polynomial with integer coefficients.

Definition 5.7. $z \in \mathbb{C}$ is an **algebraic integer** if z is an eigenvalue of a matrix with integer entries.

Proposition 5.8. *Definitions (5.6) and (5.7) are equivalent.*

Proof. To show (5.7) $\Rightarrow$ (5.6), notice that z is a root of the characteristic polynomial of the matrix (a monic polynomial with integer coefficients).
To show (5.6) $\Rightarrow$ (5.7), suppose z is a root of

$$p(x) = x^n + a_1 x^{n-1} + \ldots + a_{n-1}x + a_n.$$

Then the characteristic polynomial of the following matrix is $p(x)$:

$$\begin{pmatrix} 0 & 0 & 0 & \ldots & 0 & -a_n \\ 1 & 0 & 0 & \ldots & 0 & -a_{n-1} \\ 0 & 1 & 0 & \ldots & 0 & -a_{n-1} \\ & & & \vdots & & \\ 0 & 0 & 0 & \ldots & 1 & -a_1 \end{pmatrix}.$$

Since z is a root of the characteristic polynomial of this matrix, it is its eigenvalue. ☐

We will sometimes be using the symbol $\mathbb{A}$ to denote the set of algebraic integers.

Proposition 5.9. $\mathbb{A}$ *is a ring.*

Proof. We will be using definition (5.7). Let α be an eigenvalue of

$$\mathcal{A} \in \mathrm{Mat}_n(\mathbb{C})$$

with eigenvector v, let β be an eigenvalue of

$$\mathcal{B} \in \mathrm{Mat}_m(\mathbb{C})$$

with eigenvector w. Then $\alpha + \beta$ is an eigenvalue of

$$\mathcal{A} \otimes \mathrm{Id}_m + \mathrm{Id}_n \otimes \mathcal{B},$$

and $\alpha\beta$ is an eigenvalue of

$$\mathcal{A} \otimes \mathcal{B}.$$

The corresponding eigenvector is in both cases $v \otimes w$. ☐

Proposition 5.10. $\mathbb{A} \cap \mathbb{Q} = \mathbb{Z}$.

Proof. We will be using definition (5.6). Let z be a root of

$$p(x) = x^n + a_1 x^{n-1} + \ldots + a_{n-1}x + a_n,$$

and suppose

$$z = \frac{p}{q} \in \mathbb{Q}, gcd(p, q) = 1.$$

Notice that the leading term of $p(x)$ will have q^n in the denominator, whereas all the other terms will have a lower power of q there. Thus, if

$$q \neq \pm 1,$$

then

$$p(z) \notin \mathbb{Z},$$

a contradiction. Thus,

$$z \in \mathbb{A} \cap \mathbb{Q} \Rightarrow z \in \mathbb{Z}.$$

The reverse inclusion follows because $n \in \mathbb{Z}$ is a root of $x - n$. ☐

5.4 Frobenius divisibility

Theorem 5.11. *Let G be a finite group, and let V be an irreducible (necessarily finite-dimensional) representation of G over $\mathbb{C}$. Then*

$$|G| \text{ divides } \dim V.$$

Proof. Let

$$C_1, C_2, \ldots, C_n$$

be the conjugacy classes of G, with

$$C_1 = \{e\}.$$

Let

$$p_{C_i} \in \mathbb{C}[G]$$

be defined for each conjugacy class as

$$p_{C_i} = \sum_{g \in C_i} g.$$

Since G acts transitively on each conjugacy class, every conjugate of p_{C_i} is equal to itself, i.e. p_{C_i} is a central element in $\mathbb{C}[G]$. By Schur's lemma, p_{C_i} acts on V by a scalar λ_i; therefore,

$$|C_i|\chi_V(g_{C_i}) = \operatorname{tr}(p_{C_i}) = \dim V \cdot \lambda_i.$$

Therefore,

$$\lambda_i = \chi_V(g_{C_i})\frac{|C_i|}{\dim V},$$

where g_{C_i} is a representative of C_i. **Claim.** The number λ_i is an algebraic integer for all i.

Proof of claim. Notice that

$$p_{C_i}p_{C_j} = \sum_{g \in C_i, h \in C_j} gh = \sum_{u \in G} ux,$$

where x is the number of ways to obtain u as gh for some $g \in C_i, h \in C_j$.
Thus,

$$p_{C_i}p_{C_j} = \sum_{i,j,k} N_{ij}^k p_{C_k},$$

where N_{ij}^k is the number of ways to obtain some element of C_k as gh for some $g \in C_i, h \in C_j$.
Therefore,

$$\lambda_i\lambda_j = \sum_{i,j,k} N_{ij}^k \lambda_k$$

Let $\vec{\lambda} = \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_n \end{pmatrix}$,

then

$$N_i(\vec{\lambda}) = \lambda_i \vec{\lambda},$$

where N_i is the matrix whose jk^{th} entry is N_{ij}^k. Since

$$\vec{\lambda} \neq 0,$$

λ_i is an eigenvalue of an integer matrix N_i, and by definition (5.7) an algebraic integer. $\square$

Now, consider

$$\sum_i \lambda_i \overline{\chi_V(g_{C_i})}.$$

This is an algebraic integer, since λ_i was just proven to be an algebraic integer, and $\chi_V(g_{C_i})$ is a sum of roots of unity (it is the sum of eigenvalues of the matrix of $\rho(g_{C_i})$, and since

$$g_{C_i}^{|G|} = e$$

in G, the eigenvalues of $\rho(g_{C_i})$ are roots of unity), and $\mathbb{A}$ is a ring (5.9). On the other hand, from the definition of λ_i,

$$\sum_{C_i} \lambda_i \overline{\chi_V(g_{C_i})} = \sum_i \frac{|C_i|\chi_V(g_{C_i})\overline{\chi_V(g_{C_i})}}{\dim V}.$$

Recalling that χ_V is a class function, this is equivalent to

$$\sum_{g \in G} \frac{\chi_V(g)\overline{\chi_V(g)}}{\dim V} = \frac{|G|(\chi_V, \chi_V)}{\dim V}.$$

Since V was an irreducible representation,

$$(\chi_V, \chi_V) = 1,$$

so

$$\sum_{C_i} \lambda_i \overline{\chi_V(g_{C_i})} = \frac{|G|}{\dim V}.$$

Since

$$\frac{|G|}{\dim V} \in \mathbb{Q}$$

and

$$\sum_{C_i} \lambda_i \overline{\chi_V(g_{C_i})} \in \mathbb{A},$$

by (5.10)

$$\frac{|G|}{\dim V} \in \mathbb{Z}.$$

$\square$

5.5 Burnside's Theorem

This famous result in group theory was proved by the British mathematician William Burnside in the late 19th century. Here is a proof of his theorem using Representation Theory.

Definition 5.12. A group G is called *solvable* if there exists a series of nested normal subgroups

$$\{e\} = G_1 \triangleleft G_2 \triangleleft \dots \triangleleft G_n = G$$

where G_{i+1}/G_i is abelian for all $1 \leq i \leq n-1$.

Remark 5.13. These groups are called solvable because they first arose as Galois groups of polynomial equations which are solvable in radicals.

Theorem 5.14 (Burnside)**.**

Any group G of order $p^a q^b$, where p and q are prime and $a, b \geq 0$, is solvable.

Before proving Burnside's theorem we will prove several other results which may be of independent interest.

Theorem 5.15. *Let V be an irreducible representation of a finite group G and let C be a conjugacy class of G with* $\gcd(|C|, \dim(V)) = 1$.
Then for any $g \in C$, either $\chi_V(g) = 0$ or g acts as a scalar on V.

The proof will be based on the following two lemmas.

The first lemma is a standard fact about algebraic numbers.

Lemma 5.16. *Let x, y be algebraic numbers. Then any conjugate of $x + y$ can be expressed as $x_i + y_j$, where x_i and y_j are conjugates of x and y respectively.*

Proof. Recall that, by definition, a conjugate of an algebraic number is any root of its minimal polynomial (thus any number is its own conjugate).

Let $\{x_1, \dots, x_m\}$ and $\{y_1, \dots, y_n\}$ be the sets of all conjugates of x and y respectively. Consider the polynomial

$$g(z) = \prod_{i,j} (z - x_i - y_j) = g_0 z^{mn} + g_1 z^{mn-1} + \dots + g_{mn}.$$

The coefficients g_k depend polynomially on x_i and y_j. Moreover, these polynomials are symmetric with respect to $\{x_i\}$ and with respect to $\{y_j\}$ and have rational coefficients. Therefore all $g_k \in \mathbb{Q}$.

Notice that $x + y$ is a root of g. As we have shown, $g \in \mathbb{Q}[z]$, so the minimal polynomial of $x+y$ divides g. Therefore, every conjugate of $x+y$ is a root of g and can be written as $x_i + y_j$. □

Lemma 5.17. *If $\epsilon_1, \epsilon_2 \dots \epsilon_n$ are roots of unity such that $\dfrac{1}{n}(\epsilon_1 + \epsilon_2 + \dots + \epsilon_n)$ is an algebraic integer, then either $\epsilon_1 = \dots = \epsilon_n$ or $\epsilon_1 + \dots + \epsilon_n = 0$.*

Proof. Let $a = \frac{1}{n}(\epsilon_1 + \dots + \epsilon_n)$. Let $q = x^m + q_{m-1}x^{m-1} + \dots + q_1 x + q_0$ be the minimal polynomial of a, and let $\{a_i\}, i = 1, \dots, m$ be the set of all the conjugates of a.

By 5.16, $a_i = \frac{1}{n}(\epsilon'_1 + \epsilon'_2 + \dots + \epsilon'_n)$, where ϵ'_i are conjugate to ϵ_i. Since conjugates of roots of unity are roots of unity, $|\epsilon'_i| = 1$. This means that $|\epsilon'_1 + \dots + \epsilon'_n| \leq n$ and $|a_i| \leq 1$. Let $q = x^n + q_1 x^{n-1} + .. + q_n x$ be the minimal polynomial of a. Then, $|q_0| = \prod_{i=1}^n |a_i| \leq 1$. However, by our assumption, a is an algebraic integer and $q_0 \in \mathbb{Z}$. Therefore, either $q_0 = 0$ or $|q_0| = 1$.

Assume that $|q_0| = 1$. Then $|a_i| = 1$ for all i, and in particular, $|a| = |\frac{1}{n}(\epsilon_1 + \ldots + \epsilon_n)| = 1 = \frac{1}{n}(|\epsilon_1| + |\epsilon_2| + \ldots + |\epsilon_n|)$. This means that all ϵ_i have the same argument. It follows that $\epsilon_1 = \ldots = \epsilon_n$ since all ϵ_i have the same absolute value.

Otherwise, $q_0 = 0$. Then $x|q$ and since q is irreducible, $q = x$ and $a = 0$. □

Proof of theorem 5.15.

Let $\dim V = n$. Let $\epsilon_1, \epsilon_2, \ldots \epsilon_n$ be eigenvalues of $\rho(g)$. Since G is a finite group, $\rho(g)$ is diagonalizable and ϵ_i are roots of unity. We know that $\frac{1}{n}(|C|\chi_V(g)) \in \mathbb{A}$ and that $\chi_V(g) \in \mathbb{A}$. Since $GCD(n, |C|) = 1$, there are integers α, β such that $\alpha n + \beta|C| = 1$. Therefore,

$$\alpha\chi_V(g) + \beta\frac{|C|\chi_V(g)}{n} = \frac{\chi_V(g)}{n} \in \mathbb{A}.$$

However, since $\chi_V(g) = \epsilon_1 + \ldots + \epsilon_n$, we get that either $\epsilon_1 + \ldots + \epsilon_n = \chi_V(g) = 0$ or $\epsilon_1 = \ldots = \epsilon_n$ by 5.17. If $\epsilon_1 = \ldots = \epsilon_n$, then, since $\rho(g)$ is diagonalizable, it must be scalar. Otherwise, $\chi_V(G) = 0$. □

Proposition 5.18. *Let G be a finite, simple non-abelian group and let V be a non-trivial, irreducible representation of G. Then, if $g \in G$ acts by a scalar in V, $g = e$.*

Proof. Assume that $g \neq e$. Let N be the set of all $x \in G$ whose action in V is scalar. Clearly, $N \triangleleft G$ and $g \in N$. Since $g \neq e$, this means that $N \neq e$ and $N = G$.

Now let K be the kernel of $\rho : G \to \mathrm{End}V$. Since ρ is a group homomorphism, $K \triangleleft G$ and because V is non-trivial, $K \neq G$ and $K = \{e\}$.

This means that ρ is an injection and $G \cong \mathrm{Im}\rho$. But $\rho(x)$ is scalar for any $x \in G$, so G is commutative, which is a contradiction.

□

We are now ready to prove another result in group theory which will later imply Burnside's Theorem.

Theorem 5.19. *Let G be a group and let C be a conjugacy class of order p^k where p is prime and $k > 0$. Then G has a proper normal subgroup.*

Proof. Assume the contrary, i.e. that G is simple.

We can choose an element $e \neq g \in C$.

Let R be the regular representation of G. Since $g \neq e$, $\chi_R(g) = 0$. On the other hand, $R = \bigoplus_{V \in X} (\dim V)V$, where X is the set of all irreducible representations of G. Therefore

$$0 = \chi_R(g) = \sum_{V \in X} \dim V \chi_V(g).$$

We can divide X into three parts:

1. S, the set of irreducible representations whose dimension is divisible by p,

2. T, the set of non-trivial representations whose dimension is not divisible by p and

3. I, the trivial representation.

Lemma 5.20. *If $V \in T$ then $\chi_V(g) = 0$.*

Proof. Since $\gcd(|C|, dim(V)) = 1$, by 5.15, either

1. $\chi_V(g) = 0$ or

2. g acts as a scalar in V, and by 5.18, $g = e$ which is a contradiction.

□

Also, if $V \in S$, we have $\frac{1}{p}\dim(V)\chi_V(g) \in \mathbb{A}$, so

$$a = \sum_{V \in S} \frac{1}{p}\dim(V)\chi_V(g) \in \mathbb{A}.$$

Therefore,

$$0 = \sum_{V \in S} \dim V \chi_V(g) + \sum_{V \in T} \dim V \chi_V(g) + \dim I \chi_I(g) = pa + 1.$$

This means that $a = \frac{-1}{p}$ which is not an algebraic integer, so we have a contradiction.

□

Now we can finally prove Burnside's theorem.

Assume that there exists a group of order $p^a q^b$ that is not solvable. We may assume that G has the smallest order among such groups. Since $|G| \neq 1$, either a or b must be non-zero.

We may assume without loss of generality that $b \neq 0$.

Lemma 5.21. *Let G be a group as above.*

(i) G is simple.

(ii) G has a trivial center (in particular, it is not abelian).

(iii) G has a conjugacy class C of order p^k.

Proof.

(i) Assume that N is a non-trivial proper normal subgroup of G. Since $|N|$ divides $|G|$, $|N| = p^r q^s$ for some $r \leq a, s \leq b$.

Let $H = N/G$. Then $|H| = p^{a-r}q^{b-s}$. By our minimality assumption, both N and H are solvable, and there exist normal series

$$\{e\} = N_1 \triangleleft N_2 \triangleleft \ldots \triangleleft N_m = N \text{ and } \{e\} = H_1 \triangleleft H_2 \triangleleft \ldots \triangleleft H_n = H$$

with abelian quotients.

Let π be the canonical epimorphism $G \to G/N = H$. Then

$$\pi^{-1}(H_j) \triangleleft \pi^{-1}(H_{j+1}) \text{ and } \pi^{-1}(H_{j+1})/\pi^{-1}(H_j) = H_{j+1}/H_j$$

for any j.

Consider the normal series

$$\{e\} \triangleleft N_1 \triangleleft \ldots \triangleleft N_m = N = \pi^{-1}(e) = \pi^{-1}(H_1) \triangleleft \ldots \triangleleft \pi^{-1}(H_n) = G.$$

The quotient of any two consecutive subgroups of this series is either N_{i+1}/N_i or $\pi^{-1}(H_{j+1})/\pi^{-1}(H_j) = H_{j+1}/H_j$, all of which are abelian. Because of this, G is solvable, which is a contradiction. Therefore G is simple.

(ii) The center of G, $Z(G)$ is a normal subgroup of G. If $Z(G) = G$ then G is abelian and $\{e\} \triangleleft G$ is a normal series with abelian factors. Therefore, since G is simple, $Z(G) = \{e\}$.

(iii) Assume that G does not have a conjugacy class of order p^k. Let C be any conjugacy class. The order of C divides $|G|$, so $|C| = p^i q^j$. Because of our assumption, either $j \neq 0$ and q divides $|C|$ or $|C| = 1$ and C is central. But there is exactly one central element, e, so $|G| = \sum |C| = 1 \mathrm{mod} q$ where the sum is taken over all the conjugacy classes C. However, since $b \neq 0$, q divides $|G|$, which is a contradiction.

□

But by Theorem 5.19, this is impossible! Therefore, there are no groups of order $p^a q^b$ which are not solvable, and we have proven Burnside's Theorem.

5.6 Induced Representations

Given a representation V of a group G and a subgroup $H < G$, there is a natural way to construct a representation of H. The restricted representation of V to H, $\mathrm{Res}_H^G V$ is the representation given by the vector space V and the action $\rho_{\mathrm{Res}_H^G V} = \rho_V|_H$.

There is also a natural, but more complicated way to construct a representation of a group G given a representation V of its subgroup H.

5.6.1 Definition

Definition 5.22. If G is a group, $H < G$, and V is a representation of H, then the *induced representation* $Ind_H^G V$ is a representation of G with

$$\mathrm{Ind}_H^G V = \{f : G \to V | f(hx) = \rho_V(h) f(x)\} \forall x \in G, h \in H$$

and the action $g(f)(x) = f(xg) \forall g \in G$.

5.6.2

Let us check that this is indeed a representation:

$g(f)(hx) = f(hxg) = \rho_V(h)f(xg) = \rho_V(h)g(f)(x)$, and $g(g'(f))(x) = g'(f)(xg) = f(xgg') = (gg')(f)(x)$ for any $g, g', x \in G$ and $h \in H$.

Remark 5.23. In fact, $\mathrm{Ind}_H^G V$ is naturally equivalent to $\mathrm{Hom}_H(k[G], V)$.

Remark 5.24. Notice that if we choose a representative x_σ from every left H-coset σ of G, then any $f \in \mathrm{Ind}_H^G V$ is uniquely determined by $\{f(x_\sigma)\}$.

Because of this,

$$dim(\mathrm{Ind}_H^G V) = dim V \cdot \frac{|G|}{|H|}.$$

5.6.3 The character of induced representation

Let us now compute the character χ of $\mathrm{Ind}_H^G V$.

For a left H-coset of G, σ let us define

$$V_\sigma = \{f \in \mathrm{Ind}_H^G V | f(g) = 0 \ \forall g \ \notin \ \sigma\}.$$

Theorem 5.25. *(The Mackey formula) One has*

$$\chi(g) = \frac{1}{|H|} \sum_{x \in G, xgx^{-1} \in H} \chi_V(xgx^{-1})$$

Proof. One has

$$\mathrm{Ind}_H^G V = \bigoplus_\sigma V_\sigma,$$

and so

$$\chi(g) = \sum_\sigma \chi_\sigma(g),$$

where $\chi_\sigma(g)$ is the trace of the projection of $\rho(g)|_{V_\sigma}$ onto V_σ.

Since $g(\sigma) = \sigma g$ is a left H-coset for any left H-coset σ, $\chi_\sigma(g) = 0$ if $\sigma \neq \sigma g$.

Now assume that $\sigma = \sigma g$. Choose $x_\sigma \in \sigma$. Then $x_\sigma g = h x_\sigma$ where $h = x_\sigma g x_\sigma^{-1} \in H$. Consider the vector space homomorphism $\alpha : V_\sigma \to V$ with $\alpha(f) = f(x_\sigma)$. Since $f \in V_\sigma$ is uniquely determined by $f(x_\sigma)$, α is an isomorphism. We have

$$\alpha(gf) = g(f)(x_\sigma) = f(x_\sigma g) = f(hx_\sigma) = \rho_V(h)f(x_\sigma) = h\alpha(f),$$

and $gf = \alpha^{-1}h\alpha(f)$. This means that $\chi_\sigma(g) = \chi_V(h)$. Therefore

$$\chi(g) = \sum_{\sigma \in G \setminus H, \sigma g = \sigma} \chi_V(x_\sigma g x_\sigma^{-1}).$$

Since it does not matter which representative x_σ of σ we choose, this expression can be simplified to the statement of the theorem.

□

5.6.4 Frobenius reciprocity

A very important result about induced representations is the Frobenius Reciprocity Theorem which connects the operators Ind and Res. On the language of category theory, it states that Res and Ind are adjoint functors.

Theorem 5.26. (Frobenius Reciprocity) Let $H < G$ be groups, V be a representation of G and W a representation of H. Then $\mathrm{Hom}_G(V, \mathrm{Ind}_H^G W)$ is naturally isomorphic to $\mathrm{Hom}_H(\mathrm{Res}_H^G V, W)$.

Proof. Let $A = \mathrm{Hom}_G(V, \mathrm{Ind}_H^G W)$ and $B = \mathrm{Hom}_H(\mathrm{Res}_H^G V, W)$. Define $F : A \to B$ and $F^{-1} : B \to A$ as follows: $F(\alpha)v = \alpha v(e)$ for any $\alpha \in A$ and $F^{-1}(\beta)v(x) = \beta(xv)$ for any $\beta \in B$. □

In order to check that F and F^{-1} are well defined and inverse to each other, we need to check the following five statements.

Let $\alpha \in A$, $\beta \in B$, $v \in V$, and $x, g \in G$.

(a) $F(\alpha)$ is an H-homomorphism, i.e. $F(\alpha)(hv) = hF(\alpha)v$.

Indeed, $F(\alpha)(hv) = (\alpha(hv))(e) = h(\alpha v)(e) = (\alpha v)(h) = (h(\alpha v))(e) = hF(\alpha)v$.

(b) $(F^{-1}(b))v \in \mathrm{Ind}_H^G W$, i.e. $F^{-1}(\beta)(v)(hx) = h((F^{-1}(\beta)(v))(x))$

Indeed, $F^{-1}(\beta)(v)(hx) = \beta(hxv) = h(\beta(x_v)) = h((F^{-1}(\beta)(v))(x))$.

(c) F^{-1} is a G-homomorphism, i.e. $(F^{-1}(\beta)(gv))(x) = (gF^{-1}(\beta)(v))(x)$.

Indeed, $(F^{-1}(\beta)(gv)) = \beta(x(gv)) = \beta((xg)v) = (F^{-1}(b)(v))(xg) = (gF^{-1}(\beta)(v))(x)$.

(d) $F \circ F^{-1} = Id_B$.

This holds since $F(F^{-1}(\beta)(v)) = (F^{-1}(b)(v))(1) = \beta(v)$.

(e) $F^{-1} \circ F = Id_A$, i.e. $(F^{-1}(F(\alpha))(v))(x) = (\alpha(v))(x)$.

Indeed, $(F^{-1}(F(\alpha))(v))(x) = F(\alpha(xv)) = (\alpha(xv))(1) = (x(\alpha(v)))(1) = (\alpha(v))(x)$, and we are done.

5.6.5 Examples

Here are some examples of Induced Representations. (we use the notation for representations from the character tables)

1. Let $G = S_3$, $H = \mathbb{Z}_2$. Using the Frobenius reciprocity, we obtain: $\mathrm{Ind}_H^G\mathbb{C} = \mathbb{C}^2 \oplus \mathbb{C}$, $\mathrm{Ind}_H^G\mathbb{C}_- = \mathbb{C}^2 \oplus \mathbb{C}_-$.
2. Let $G = S_3$, $H = \mathbb{Z}_3$. Then we obtain $\mathrm{Ind}_H^G\mathbb{C} = \mathbb{C} \oplus \mathbb{C}_-$, $\mathrm{Ind}_H^G\mathbb{C}_\epsilon = \mathrm{Ind}_H^G\mathbb{C}_{\epsilon^2} = \mathbb{C}^2$.
3. Let $G = S_4$, $H = S_3$. Then $\mathrm{Ind}_H^G\mathbb{C} = \mathbb{C} \oplus \mathbb{C}_-^3$, $\mathrm{Ind}_H^G\mathbb{C}_- = \mathbb{C}_- \oplus \mathbb{C}_+^3$, $\mathrm{Ind}_H^G\mathbb{C}^2 = \mathbb{C}^2 \oplus \mathbb{C}_-^3 \oplus \mathbb{C}_+^3$.

5.7 Representations of S_n

Let us give, without proof, a description of the representations of the symmetric group S_n for any n.

Let λ be a partition of n, i.e. a representation of n in the form $n = \lambda_1 + \lambda_2 + ... + \lambda_p$, where λ_i are positive integers, and $\lambda_i \geq \lambda_{i+1}$ To such λ we will attach a Young diagram Y_λ, which is the subset of the coordinate plane defined by the inequalities: $0 \leq x \leq p$, $0 \leq y \leq \lambda_{[x]_+}$, where $[x]_+$ denotes the smallest integer N such that $N \geq x$. Clearly, Y_λ is a collection of n unit squares. A Young tableau corresponding to Y_λ is the result of filling numbers $1, ..., n$ into the squares iof Y_λ in some way (without repretitions). For example, we will consider the Young tableau T_λ obtained by filling in the numbers in the increasing order, left to right, bottom to top.

We can define two subgroups of S_n corresponding to Y_λ:

1. Row subgroup P_λ: the subgroup which maps every element of $\{1, ..., n\}$ to an element standing in the same row in T_λ.

2. Column subgroup P_λ: the subgroup which maps every element of $\{1, ..., n\}$ to an element standing in the same column in T_λ.

Clearly, $P \cap Q = \{1\}$.

Define the *Young projectors:*

$$p_\lambda^+ := \frac{1}{|P_\lambda|}\sum_{g \in P_\lambda} g,$$

$$q_\lambda^- := \frac{1}{|Q_\lambda|}\sum_{g \in Q_\lambda} (-1)^g g,$$

where $(-1)^g$ denotes teh sign of the permutation g. Set $c_\lambda = p_\lambda q_\lambda$.

The irreducible representations of S_n are described by the following theorem.

Theorem 5.27. *The subspace $V_\lambda := \mathbb{C}[S_n]c_\lambda$ of $\mathbb{C}[S_n]$ is an irreducible representation of S_n under left multiplication. Every irreducible representation of S_n is isomorphic to V_λ for a unique λ.*

Example 5.28.

For the partition $\lambda = (1, ..., 1)$, $P_\lambda = S_n$, $Q_\lambda = \{1\}$, so c_λ is the symmetrizer, and hence V_λ is the trivial representation.

For the partition $\lambda = (n)$, $Q_\lambda = S_n$, $P_\lambda = \{1\}$, so c_λ is the antisymmetrizer, and hence V_λ is the sign representation.

$n = 3$. For $\lambda = (2,1)$, $V_\lambda = \mathbb{C}^2$.

$n = 4$. For $\lambda = (2,2)$, $V_\lambda = \mathbb{C}^2$; for $\lambda = (3,1)$, $V_\lambda = \mathbb{C}^3_+$; for $\lambda = (2,1,1)$, $V_\lambda = \mathbb{C}^3_-$.

Corollary 5.29. *All irreducible representations of S_n can be given by matrices with rational entries.*

Problem 5.30. *Find the sum of dimensions of all irreducible representations of the symmetric group S_n.*

Hint. Show that all irreducible representations of S_n are real, i.e. admit a nondegenerate invariant symmetric form. Then use the Frobenius-Schur theorem.

6 Representations of $GL_2(\mathbb{F}_q)$

6.1 Conjugacy classes in $GL_2(\mathbb{F}_q)$

Let $\mathbb{F}_q$ be a finite field of size q of characteristic other than 2. Then

$$|GL_2(\mathbb{F}_q)| = (q^2 - 1)(q^2 - q),$$

since the first column of an invertible 2 by 2 matrix must be non-zero and the second column may not be a multiple of the first one. Factoring,

$$|GL_2(\mathbb{F}_q)| = q(q+1)(q-1)^2.$$

The goal of this section is to describe the irreducible representations of $GL_2(\mathbb{F}_q)$.
To begin, let us find the conjugacy classes in $GL_2(\mathbb{F}_q)$.

Representatives	Number of elements in a conjugacy class	Number of classes
Scalar $\left(\begin{smallmatrix} x & 0 \\ 0 & x \end{smallmatrix}\right)$	1 (this is a central element)	$q-1$ (one for every non-zero x)
Parabolic $\left(\begin{smallmatrix} x & 1 \\ 0 & x \end{smallmatrix}\right)$	q^2-1 (elements that commute with this one are of the form $\left(\begin{smallmatrix} t & u \\ 0 & t \end{smallmatrix}\right)$, $t \neq 0$)	$q-1$ (one for every non-zero x)
Hyperbolic $\left(\begin{smallmatrix} x & 0 \\ 0 & y \end{smallmatrix}\right)$, $y \neq x$	q^2+q (elements that commute with this one are of the form $\left(\begin{smallmatrix} t & 0 \\ 0 & u \end{smallmatrix}\right)$, $t, u \neq 0$)	$\frac{1}{2}(q-1)(q-2)$ ($x, y \neq 0$ and $x \neq y$)
Elliptic $\left(\begin{smallmatrix} x & \epsilon y \\ y & x \end{smallmatrix}\right)$, $x \in \mathbb{F}_q$, $y \in \mathbb{F}_q^\times$, $\epsilon \in \mathbb{F}_q \setminus \mathbb{F}_q^2$ (characteristic polynomial over $\mathbb{F}_q$ is irreducible)	$q^2 - q$ (the reason will be described below)	$\frac{1}{2}q(q-1)$ (matrices with y and $-y$ are conjugate)

More on the conjugacy class of elliptic matrices: these are the matrices whose characteristic polynomial is irreducible over $\mathbb{F}_q$ and which therefore don't have eigenvalues in $\mathbb{F}_q$. Let A be such a matrix, and consider a quadratic extension of $\mathbb{F}_q$,

$$\mathbb{F}_q(\sqrt{\epsilon}), \epsilon \in \mathbb{F}_q \setminus \mathbb{F}_q^2.$$

Over this field, A will have eigenvalues

$$\alpha = \alpha_1 + \sqrt{\epsilon}\alpha_2$$

and

$$\overline{\alpha} = \alpha_1 - \sqrt{\epsilon}\alpha_2,$$

with corresponding eigenvectors

$$v,\ \overline{v} \quad (Av = \alpha v,\ A\overline{v} = \overline{\alpha v}).$$

Choose a basis

$$\{e_1 = v + \overline{v},\ e_2 = \sqrt{\epsilon}(v - \overline{v})\}.$$

In this basis, the matrix A will have the form

$$\begin{pmatrix} \alpha_1 & \epsilon\alpha_2 \\ \alpha_2 & \alpha_1 \end{pmatrix},$$

justifying the description of representative elements of this conjugacy class.
In the basis $\{v, \overline{v}\}$, matrices that commute with A will have the form

$$\begin{pmatrix} \lambda & 0 \\ 0 & \overline{\lambda} \end{pmatrix},$$

for all

$$\lambda \in \mathbb{F}_{q^2}^{\times},$$

so the number of such matrices is $q^2 - 1$.

6.2 Representations of $GL_2(\mathbb{F}_q)$

In this section, G will denote the group $GL_2(\mathbb{F}_q)$.

6.2.1 1-dimensional representations

First, we describe the 1-dimensional representations of G.

Proposition 6.1. $[G, G] = SL_2(\mathbb{F}_q)$.

Proof. Clearly,

$$\det(xyx^{-1}y^{-1}) = 1,$$

so

$$[G, G] \subseteq SL_2(\mathbb{F}_q).$$

To show the converse, let us show that every elementary matrix with determinant 1 is a commutator; such matrices generate $SL_2(\mathbb{F}_q)$. We may restruict our attention to the matrices

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

or

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

Clearly, only showing that the first matrix is a commutator suffices, since if

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = ABA^{-1}B^{-1}$$

then

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}^t = (B^t)^{-1}(A^t)^{-1}B^tA^t.$$

To show that matrix

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

is a commutator, we observe that the commutator of the following two matrices is as required:

$$A = \begin{pmatrix} k & 0 \\ 0 & \frac{1}{k} \end{pmatrix}, \ B = \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix},$$

where

$$y = \frac{1}{k^2 - 1}, \ k \notin \{0, 1, -1\}$$

(the case of $\mathbb{F}_3$ is considered below).
In the case of $\mathbb{F}_3$, for

$$A = \begin{pmatrix} 2 & 2 \\ 0 & 1 \end{pmatrix}, \ B = \begin{pmatrix} 2 & 2 \\ 0 & 2 \end{pmatrix}, \text{ we have } ABA^{-1}B^{-1} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

This completes the proof. □

Therefore,

$$G/[G,G] \cong \mathbb{F}_q^\times \text{ via } g \to \det(g).$$

The one-dimensional representations of G thus have the form

$$\rho(g) = \xi\big(\det(g)\big),$$

where ξ is a homomorphism

$$\xi : \mathbb{F}_q^\times \to \mathbb{C}^\times;$$

since $\mathbb{F}_q^\times$ is cyclic, there are $q-1$ such representations, denoted $\mathbb{C}_\xi$.

6.2.2 Principal series representations

Let

$$B \subset G, \ B = \{\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}\}$$

(the set of upper triangular matrices); then

$$|B| = (q-1)^2 q,$$

$$[B,B] = U = \{\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}\},$$

and

$$B/[B,B] \cong \mathbb{F}_q^\times \times \mathbb{F}_q^\times$$

(the isomorphism maps an element of $B/[B,B]$ to its two eigenvalues).
Let

$$\lambda : B \to \mathbb{C}^*$$

be a homomorphism defined by

$$\lambda \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \lambda_1(a)\lambda_2(c), \text{for some pair of homomorphisms } \lambda_1, \lambda_2 : \mathbb{F}_q \to \mathbb{C}.$$

Define

$$V_{\lambda_1,\lambda_2} = Ind_B^G \mathbb{C}_\lambda,$$

where $\mathbb{C}_\lambda$ is the 1-dimensional representation of B in which B acts by λ. We have

$$\dim(V_{\lambda_1,\lambda_2}) = \frac{|G|}{|B|} = q+1.$$

Theorem 6.2. *1. $\lambda_1 \neq \lambda_2 \Rightarrow V_{\lambda_1,\lambda_2}$ is irreducible.*

2. $\lambda_1 = \lambda_2 = \mu \Rightarrow V_{\lambda_1,\lambda_2} = \mathbb{C}_\mu \oplus W_\mu$, where W_μ is a q-dimensional irreducible representation of G.

3. $W_\mu \cong W_\nu$ iff $\mu = \nu$; $V_{\lambda_1,\lambda_2} \cong V_{\lambda_1,\lambda_2}$ iff $\{\lambda_1, \lambda_2\} = \{\lambda_1', \lambda_2'\}$ (in the second case, $\lambda_1 \neq \lambda_2, \lambda_1' \neq \lambda_2'$).

Proof. From the Mackey formula, we have

$$tr_{V_{\lambda_1,\lambda_2}}(g) = \frac{1}{|B|} \sum_{a \in G,\, aga^{-1} \in B} \lambda(aga^{-1}).$$

If

$$g = \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix},$$

the expression on the right evaluates to

$$\lambda_1\lambda_2(x)\frac{|G|}{|B|} = \lambda_1(x)\lambda_2(x)\big(q+1\big).$$

If

$$g = \begin{pmatrix} x & 1 \\ 0 & x \end{pmatrix},$$

the expression evaluates to

$$\lambda_1\lambda_2(x) \cdot 1,$$

since here

$$aga^{-1} \in B \Rightarrow a \in B.$$

If

$$g = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix},$$

the expression evaluates to

$$\big(\lambda_1(x)\lambda_2(y) + \lambda_1(y)\lambda_2(x)\big) \cdot 1,$$

since here

$$aga^{-1} \in B \Rightarrow a \in B \text{ or } a \text{ is an element of a permutation of } B.$$

If

$$g = \begin{pmatrix} x & \varepsilon y \\ y & x \end{pmatrix},$$

the expression on the right evaluates to 0 because matrices of this type don't have eigenvalues over $\mathbb{F}_q$.
From the definition, $\lambda_i(x)(i = 1, 2)$ is a root of unity, so

$$|G|\langle \chi_{V_{\lambda_1,\lambda_2}}, \chi_{V_{\lambda_1,\lambda_2}}\rangle = (q+1)^2(q-1) + (q^2-1)(q-1) + 2(q^2+q)\frac{(q-1)(q-2)}{2} + (q^2+q)\sum_{x\neq y}\lambda_1(x)\lambda_2(y)\overline{\lambda_1(y)\lambda_2(x)}.$$

The last two summands come from the expansion

$$|a+b|^2 = |a|^2 + |b|^2 + a\overline{b} + \overline{a}b.$$

If

$$\lambda_1 = \lambda_2 = \mu,$$

the last term is equal to

$$(q^2+q)(q-2)(q-1),$$

and the total in this case is

$$(q+1)(q-1)[(q+1)+(q-1)-2q(q-2)] = (q+1)(q-1)2q(q-1) = 2|G|,$$

so

$$\langle \chi_{V_{\lambda_1,\lambda_2}}, \chi_{V_{\lambda_1,\lambda_2}}\rangle = 2.$$

Clearly,

$$\mathbb{C}_\mu \subseteq \mathrm{Ind}_B^G \mathbb{C}_{\mu,\mu},$$

since

$$\mathrm{Hom}_G(\mathbb{C}_\mu, \mathrm{Ind}_B^G\mathbb{C}_{\mu,\mu}) = \mathrm{Hom}_B(\mathbb{C}_\mu, \mathbb{C}_\mu) = \mathbb{C} \text{ (Theorem 5.26)}.$$

$\therefore \mathrm{Ind}_B^G\mathbb{C}_{\mu,\mu} = \mathbb{C}_\mu \oplus W_\mu$; W_μ is irreducible; and the character of W_μ is different for distinct values of μ, proving that W_μ are distinct.

If $\lambda_1 \neq \lambda_2$, let $z = xy^{-1}$, then the last term of the summation is

$$(q^2+q)\sum_{x\neq y}\lambda_1(z)\overline{\lambda_2(z)} = (q^2+q)\sum_{x;z\neq 1}\frac{\lambda_1}{\lambda_2}(z) = (q^2+q)(q-1)\sum_{z\neq 1}\frac{\lambda_1}{\lambda_2}(z).$$

Since

$$\sum_{z\in\mathbb{F}_q}\frac{\lambda_1}{\lambda_2}(z) = 0,$$

being the sum of all roots of unity, the last term becomes

$$-(q^2+q)(q-1)\sum_{z\neq 1}\frac{\lambda_1}{\lambda_2}(1) = -(q^2+q)(q-1).$$

The difference between this case and the case of $\lambda_1 = \lambda_2$ is equal to

$$-(q^2+q)[(q-2)(q-1)+(q-1)] = |G|,$$

so this is an irreducible representation.

To prove the third assertion of the theorem, we note that on hyperbolic characters the function

$$\lambda_1(x)\lambda_2(y) + \lambda_1(y)\lambda_2(x)$$

determines λ_1, λ_2 up to permutation.

□

6.2.3 Complimentary series representations

Let $\mathbb{F}_{q^2} \supset \mathbb{F}_q$ be a quadratic extension $\mathbb{F}_q(\sqrt{\varepsilon}), \varepsilon \in \mathbb{F}_q \backslash \mathbb{F}_q^2$. We regard this as a 2-dimensional vector space over $\mathbb{F}_q$; then $GL_2(\mathbb{F}_q)$ is the group of linear transformations of $\mathbb{F}_{q^2}$ over $\mathbb{F}_q$. Let $K \subset GL_2(\mathbb{F}_q)$ be the cyclic group of multiplication by an element of $\mathbb{F}_q$,

$$K = \{\begin{pmatrix} x & \varepsilon y \\ y & x \end{pmatrix}\}, \ |K| = q^2 - 1.$$

For $\nu : K \to \mathbb{C}^\times$ a homomorphism, let

$$Y_\nu = \mathrm{Ind}_K^G \nu.$$

This representation, of course, is very reducible. Let us compute its character, using the Mackey formula. We get

$$\chi \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix} = q(q-1)\nu(x);$$

$$\chi(A) = 0 \text{ for } A \text{ parabolic or hyperbolic;}$$

$$\chi \begin{pmatrix} x & \varepsilon y \\ y & x \end{pmatrix} = \nu \begin{pmatrix} x & \varepsilon y \\ y & x \end{pmatrix} + \nu \begin{pmatrix} x & \varepsilon y \\ y & x \end{pmatrix}^q.$$

The last assertion is because if we regard the matrix as an element of $\mathbb{F}_{q^2}$, conjugation is an automorphism of $\mathbb{F}_{q^2}$ over $\mathbb{F}_q$, but the only nontrivial automorphism of $\mathbb{F}_{q^2}$ over $\mathbb{F}_q$ is the q^{th} power map.

We thus have

$$\mathrm{Ind}_K^G \nu^q = \mathrm{Ind}_K^G \nu$$

because they have the same character. Therefore, for $\nu^q \neq \nu$ we get $\frac{1}{2}q(q-1)$ representations. Of course, they are reducible.

Next, we look at the following tensor product:

$$W_\varepsilon \otimes V_{\alpha,\varepsilon},$$

where ε is the trivial character and W_ε is defined as in the previous section. The character of this representation is

$$\chi \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix} = q(q+1)\alpha(x);$$

$$\chi(A) = 0 \text{ for } A \text{ parabolic or elliptic;}$$

$$\chi \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} = \alpha(x) + \alpha(y).$$

Thus the "virtual representation"

$$W_\varepsilon \otimes V_{\alpha,\varepsilon}, - V_{\alpha,\varepsilon} - \mathrm{Ind}_K^G \nu$$

where α is the restriction of ν to scalars has character

$$\chi \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix} = (q-1)\alpha(x);$$

$$\chi \begin{pmatrix} x & 1 \\ 0 & x \end{pmatrix} = -\alpha(x);$$

$$\chi \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} = 0;$$

$$\chi \begin{pmatrix} x & \varepsilon y \\ y & x \end{pmatrix} = -\nu \begin{pmatrix} x & \varepsilon y \\ y & x \end{pmatrix} - \nu^q \begin{pmatrix} x & \varepsilon y \\ y & x \end{pmatrix}.$$

In all that follows, we will have $\nu^q \neq \nu$.

The following two lemmas will establish that the inner product of this character with itself is equal to 1, that its value at 1 is positive, and that the above conditions imply that it is the character of an irreducible representation of G.

Lemma 6.3. *Let χ be the character of the "virtual representation" defined above. Then*

$$\langle \chi, \chi \rangle = 1$$

and

$$\chi(1) > 0.$$

Proof.

$$\chi(1) = q(q+1) - (q+1) - q(q-1) = q - 1 > 0.$$

We now compute the inner product $\langle \chi, \chi \rangle$. Since α is a root of unity, this will be equal to

$$\frac{1}{(q-1)^2 q(q+1)} \left[(q-1)\cdot(q-1)^2\cdot 1 + (q-1)\cdot 1 \cdot (q^2-1) + \frac{q(q-1)}{2} \cdot \sum_{\zeta \text{ elliptic}} (-\nu(\zeta) - \nu^q(\zeta))\overline{(-\nu(\zeta) - \nu^q(\zeta))}\right]$$

Because ν is also a root of unity, the last term of the expression evaluates to

$$(-\nu(\zeta))(\overline{-\nu(\zeta)}) + (-\nu^q(\zeta))(\overline{-\nu^q(\zeta)}) + \sum_{\zeta \text{ elliptic}} \nu^{q-1}(\zeta) + \nu^{1-q}(\zeta).$$

The first two summands here are equal to 1. Let's evaluate the last one.

Associating the elliptic elements with $\mathbb{F}_{q^2}$ as a vector space over $\mathbb{F}_q$, we have, since $\mathbb{F}_{q^2}^\times$ is cyclic and $\nu^q \neq \nu$,

$$\sum_{\zeta \in \mathbb{F}_{q^2}^\times} \nu^{q-1}(\zeta) = \sum_{\zeta \in \mathbb{F}_{q^2}^\times} \nu^{1-q}(\zeta) = 0.$$

However, we are only interested in $\zeta \in \mathbb{F}_{q^2}^\times \setminus \mathbb{F}_q$, since we already considered the conjugacy classes of scalars. Therefore,

$$\sum_{\zeta \text{ elliptic}} \nu^{q-1}(\zeta) + \nu^{1-q}(\zeta) = 0 - \sum_{\zeta \in \mathbb{F}_q^\times} \nu^{q-1}(\zeta) + \nu^{1-q}(\zeta) = 0 - 2(q-1) = -2(q-1)$$

since $\mathbb{F}_q^\times$ is cyclic of order $q-1$. Therefore,

$$\langle \chi, \chi \rangle = \frac{1}{(q-1)^2 q(q+1)} \big((q-1) \cdot (q-1)^2 \cdot 1 + (q-1) \cdot 1 \cdot (q^2-1) + \frac{q(q-1)}{2} \cdot (2(q^2-q) - 2(q-1)) \big) = 1.$$

□

Lemma 6.4. *Let*

$$V_1, V_2, \ldots, V_m$$

be (possibly reducible) representations of a finite group G over $\mathbb{C}$, and let

$$\chi_1, \chi_2, \ldots, \chi_m$$

be the respective characters. Let

$$p_1, p_2, \ldots p_m \in \mathbb{Z}$$

(not necessarily nonnegative!), let

$$\chi = p_1 + p_2 + \cdots + p_m.$$

If $\langle \chi, \chi \rangle = 1$ and $\chi(1) > 0$, then χ is a character of an irreducible representation of G.

Proof. Let

$$W_1, W_2, \ldots, W_s$$

be the irreducible representations of G, let

$$V_i = \bigoplus_j a_{ij} W_j \ (a_{ij} \in \mathbb{Z}^+ \cup \{0\}).$$

If ξ_j are characters of W_j, then

$$\chi_i = \sum_j a_{ij} \xi_j \Rightarrow \chi = \sum_{i,j} p_i a_{ij} \xi_j = \sum q_j \xi_j,$$

where

$$q_j = \sum_i p_i a_{ij} \in \mathbb{Z}.$$

Since

$$\langle \chi, \chi \rangle = \sum_j q_j^2 = 1,$$

we must have $q_j = 0$ except for one value of $j = j_0$, with $q_{j_0} = \pm 1$. Thus,

$$\chi = \pm \xi_{j_0},$$

and since $\chi(1) > 0$ we have

$$\chi = \xi_{j_0}.$$

□

We have now shown that for any pair $(\alpha,\ \nu)$ with $\nu^q \neq \nu$ the representation with the same character as

$$W_\varepsilon \otimes V_{\lambda,\varepsilon,} - V_{\lambda,\varepsilon} - \mathrm{Ind}_K^G \nu$$

exists and is irreducible. This character is distinct for distinct pairs $(\alpha,\ \nu)$ when ν is not a q^{th} power, so there are $\frac{q(q-1)}{2}$ such representations, each of dimension $q-1$.

We have thus found $q-1$ 1-dimensional representations of G, $\frac{q(q-1)}{2}$ principal series representations, and $\frac{q(q-1)}{2}$ complimentary series representations, for a total of q^2-1 representations, i.e. the number of conjugacy classes in G. We can also check the sum of squares formula:

$$(q-1)\cdot 1^2 + (q-1)\cdot q^2 + \frac{(q-1)(q-2)}{2}\cdot (q+1)^2 + \frac{q(q-1)}{2}\cdot (q-1)^2 = (q-1)^2 q(q+1) = |G|.$$

7 Quiver Representations

7.1 Problems

Problem 7.1. Field embeddings. *Recall that $k(y_1, ..., y_m)$ denotes the field of rational functions of $y_1, ..., y_m$ over a field k.*

(a) Let $f : k[x_1, ..., x_n] \to k(y_1, ..., y_m)$ be an injective homomorphism. Show that $m \geq n$. (Look at the growth of dimensions of the spaces W_N of polynomials of degree N in x_i and their images under f as $N \to \infty$).

(b) Let $f : k(x_1, ..., x_n) \to k(y_1, ..., y_m)$ be a field embedding. Show that $m \geq n$.

Problem 7.2. Some algebraic geometry.

Let k be an algebraically closed field, and $G = GL_n(k)$ be the group of nondegenerate matrices of size n over k. An algebraic representation of G is a finite dimensional representation $\rho : G \to GL_N(k)$ such that the matrix elements $\rho_{ij}(g)$ are polynomials in g_{pq} and $1/\det(g)$. For example, the standard representation $V = k^n$ and the dual representation V^* are algebraic (show it!).

Let V be an algebraic representation of G. Show that if G has finitely many orbits on V then $\dim(V) \leq n^2$. Namely:

(a) Let $x_1, ..., x_N$ be linear coordinates on V. Let us say that a subset X of V is Zariski dense if any polynomial $f(x_1, ..., x_N)$ which vanishes on X is zero (coefficientwise). Show that if G has finitely many orbits on V then G has at least one dense orbit on V.

(b) Use (a) to construct a field embedding $k(x_1, ..., x_N) \to k(g_{pq})$, then use problem 1.

(c) generalize the result of this problem to the case when $G = GL_{n_1}(k) \times ... \times GL_{n_m}(k)$.

Problem 7.3. Dynkin diagrams.

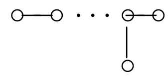
Let Γ be a graph, i.e. a finite set of points (vertices) connected with a certain number of edges. We assume that Γ is connected (any vertex can be connected to any other by a path of edges) and has no self-loops (edges from a vertex to itself). Suppose the vertices of Γ are labeled by integers $1, ..., N$. Then one can assign to Γ an $N \times N$ matrix $R_\Gamma = (r_{ij})$, where r_{ij} is the number of edges connecting vertices i and j. This matrix is obviously symmetric, and is called the adjacency matrix. Define the matrix $A_\Gamma = 2I - R_\Gamma$, where I is the identity matrix.

Main definition: Γ is said to be a Dynkin diagram if the quadratic from on $\mathbb{R}^N$ with matrix A_Γ is positive definite. Dynkin diagrams appear in many areas of mathematics (singularity theory, *Lie algebras, representation theory, algebraic geometry, mathematical physics, etc.) In this problem you will get a complete classification of Dynkin diagrams. Namely, you will prove*

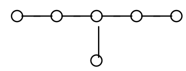
Theorem. Γ *is a Dynkin diagram if and only if it is one on the following graphs:*

- A_n :

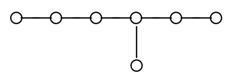
- D_n:



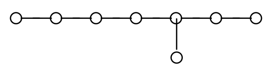
- E_6 :



- E_7 :



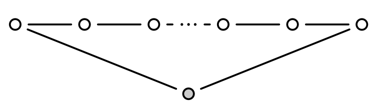
- E_8 :



(a) Compute the determinant of A_Γ *where* $\Gamma = A_N, D_N$. *(Use the row decomposition rule, and write down a recusive equation for it). Deduce by Sylvester criterion that* A_N, D_N *are Dynkin diagrams.*

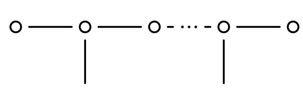
(b) Compute the determinants of A_Γ *for* E_6, E_7, E_8 *(use row decomposition and reduce to (a)). Show they are Dynkin diagrams.*

(c) Show that if Γ *is a Dynkin diagram, it cannot have cycles. For this, show that* $det(A_\Gamma) = 0$ *for a graph* Γ *below:*



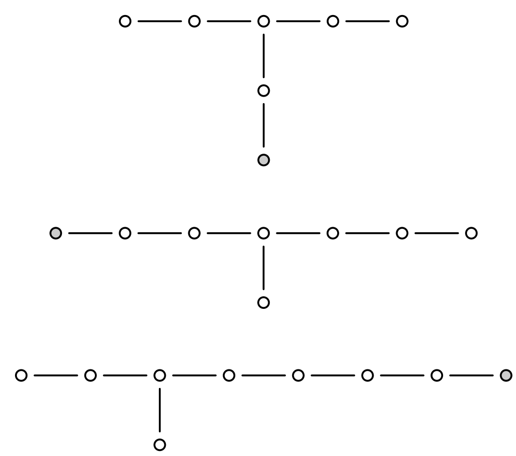
(show that the sum of rows is 0). Thus Γ *has to be a tree.*

(d) Show that if Γ *is a Dynkin diagram, it cannot have vertices with 4 or more incoming edges. For this, show that* $det(A_\Gamma) = 0$ *for a graph* Γ *below (including the case when the two nodal vertices coincide).*



(e) Using (d) show that Γ *can have no more than one vertex with 3 incoming edges.*

(f) Show that $det(A_\Gamma) = 0$ *for graphs* Γ *below:*



(g) Deduce from (a)-(f) the classification theorem for Dynkin diagrams.

(h) A (simply laced) affine Dynkin diagram is a connected graph without self-loops such that the quadratic form defined by A_Γ is positive semidefinite. Classify affine Dynkin diagrams. (Show that they are exactly the forbidden diagrams from (c)-(f)).

Problem 7.4. *Let Q be a quiver with set of vertices D. We say that Q is of finite type if it has finitely many indecomposable representations. Let b_{ij} be the number of edges from i to j in Q ($i, j \in D$).*

There is the following remarkable theorem, proved by P. Gabriel in the 1970-s.

Theorem. *A quiver Q is of finite type if and only if the corresponding unoriented graph (i.e. with directions of arrows forgotten) is a Dynkin diagram.*

In this problem you will prove the only if direction of this theorem (i.e. why other quivers are NOT of finite type).

(a) Show that if Q is of finite type then for any numbers $x_i \geq 0$ which are not simultaneously zero, one has $q(x_1, ..., x_n) > 0$, where

$$q(x_1, ..., x_n) := \sum_{i \in D} x_i^2 - \sum_{i,j \in D} b_{ij} x_i x_j.$$

Hint. It suffices to check the result for integers: $x_i = n_i$. For this, consider the space W of representations V of Q such that $\dim V_i = n_i$. Show that the group $\times_i GL_{n_i}(k)$ acts with finitely many orbits on $W \oplus k$, and use problem 2 to derive the inequality.

(b) Deduce that q is a positive definite quadratic form.

(c) Show that a quiver of finite type can have no self-loops. Then, using problem 3, deduce the theorem.

Problem 7.5. *Let G be a finite subgroup of $SU(2)$, and V be the 2-dimensional representation of G coming from its embedding into $SU(2)$. Let V_i, $i \in I$, be all the irreducible representations of G. Let r_{ij} be the multiplicity of V_i in $V \otimes V_j$.*

(a) Show that $r_{ij} = r_{ji}$.

(b) The McKay graph of G, $M(G)$, is the graph whose vertices are labeled by $i \in I$, and i is connected to j by r_{ij} edges. Show that $M(G)$ is connected. (Use problem 1)

(c) Show that $M(G)$ is an affine Dynkin graph. For this, show that the matrix $a_{ij} = 2\delta_{ij} - r_{ij}$ is positive semidefinite but not definite, and use problem set 5.

Hint. Let $f = \sum x_i \chi_{V_i}$, where χ_{V_i} be the characters of V_i. Show directly that $((2-\chi_V)f, f) \geq 0$. When is it $= 0$? Next, show that $M(G)$ has no self-loops, by using that if G is not cyclic then G contains the central element $-Id \in SU(2)$.

(d) Which groups from problem 2 correspond to which diagrams?

(e) Using the McKay graph, find the dimensions of irreducible representations of all finite $G \subset SU(2)$. Compare with the results on subgroups of $SO(3)$ we obtained earlier.

One central question when looking at representations of quivers is whether a certain quiver has only finitely many indecomposable represention s. We already proved that only those quives whose underlying undirected graph is a Dynkin diagram may have this property. To see if they actually do have this property, we first explicitly decompose representations of certain easy quivers.

7.2 Indecomposable representations of the quivers A_1, A_2, A_3

Example 7.6 (A_1)**.** The quiver A_1 consists of a single vertex and has no edges. Since a representation of this quiver is just a single vector space, the only indecomposable representation is the ground field itself. Therefore the quiver A_1 has only one indecomposable representation, namely the field of complex numbers.

Example 7.7 (A_2)**.** The quiver A_2 consists of two vertices connected by a single edge.

$$\bullet \longrightarrow \bullet$$

A representation of this quiver consists of two vector spaces V, W and an operator $A : V \to W$.

$$\underset{V}{\bullet} \xrightarrow{A} \underset{W}{\bullet}$$

To decompose this representation, we first let V' be a complement to the kernel of A in V and let W' be a complement to the image of A in W. Then we can decompose the representation as follows

$$\underset{V}{\bullet} \xrightarrow{A} \underset{W}{\bullet} = \underset{\ker V}{\bullet} \xrightarrow{0} \underset{0}{\bullet} \oplus \underset{V'}{\bullet} \xrightarrow[\sim]{A} \underset{\mathrm{Im} A}{\bullet} \oplus \underset{0}{\bullet} \xrightarrow{0} \underset{W'}{\bullet}$$

The first summand is a direct sum of objects of the type $1 \longrightarrow 0$, the second a multiple of $1 \longrightarrow 1$, the third of $0 \longrightarrow 1$. We see that the quiver A_2 has three indecomposable representations, namely [2]

$$1 \longrightarrow 0, \quad 1 \longrightarrow 1 \quad \text{and} \quad 0 \longrightarrow 1 .$$

Example 7.8 (A_3)**.** The quiver A_3 consists of three vertices and two connections between them. So we have to choose between two possible orientations.

$$\bullet \longrightarrow \bullet \longrightarrow \bullet \quad \text{or} \quad \bullet \longrightarrow \bullet \longleftarrow \bullet$$

[2] By an object of the type $1 \longrightarrow 0$ we mean a map from a one-dimensional vector space to the zero space. Similarly, an object of the type $0 \longrightarrow 1$ is a map from the zero space into an one-dimensional space. The object $1 \longrightarrow 1$ means an isomorphism from a one-dimensional to another one-dimensional space. Similarly, numbers in such diagrams always mean the dimension of the attached spaces and the maps are the canonical maps (unless specified otherwise).

1. We first look at the orientation

$$\bullet \longrightarrow \bullet \longrightarrow \bullet\ .$$

Then a representation of this quiver looks like

$$\underset{V}{\bullet} \xrightarrow{\quad A \quad} \underset{W}{\bullet} \xrightarrow{\quad B \quad} \underset{Y}{\bullet}\ .$$

Like in 7.7 we first split away

$$\underset{\ker A}{\bullet} \xrightarrow{\quad 0 \quad} \underset{0}{\bullet} \xrightarrow{\quad 0 \quad} \underset{0}{\bullet}\ .$$

This object is a multiple of $1 \longrightarrow 0 \longrightarrow 0$. Next, let Y' be a complement of $\mathrm{Im} B$. Then we can also split away

$$\underset{0}{\bullet} \xrightarrow{\quad 0 \quad} \underset{0}{\bullet} \xrightarrow{\quad 0 \quad} \underset{Y'}{\bullet}$$

which is a multiple of the object $0 \longrightarrow 0 \longrightarrow 1$. This results in a situation where the map A is injective and the map B is surjective (we rename the spaces to simplify notation):

$$\underset{V}{\bullet} \overset{A}{\hookrightarrow} \underset{W}{\bullet} \overset{B}{\twoheadrightarrow} \underset{Y}{\bullet}\ .$$

Next, let $X = \ker(B \circ A)$ and let X' be a complement of X in V. Let W' be a complement of $A(X)$ in W such that $A(X') \subset W'$. Then we get

$$\underset{V}{\bullet} \overset{A}{\hookrightarrow} \underset{W}{\bullet} \overset{B}{\twoheadrightarrow} \underset{Y}{\bullet} \;=\; \underset{X}{\bullet} \xrightarrow{\quad A \quad} \underset{A(X)}{\bullet} \xrightarrow{\quad B \quad} \underset{0}{\bullet} \;\oplus\; \underset{X'}{\bullet} \overset{A}{\hookrightarrow} \underset{W'}{\bullet} \overset{B}{\twoheadrightarrow} \underset{Y}{\bullet}$$

The first of these summands is a multiple of $1 \xrightarrow{\sim} 1 \longrightarrow 0$. Looking at the second summand, we now have a situation where A is injective, B is surjective and furthermore $\ker(B \circ A) = 0$. To simplify notation, we redefine

$$V = X',\ W = W'.$$

Next we let $X = \mathrm{Im}(B \circ A)$ and let X' be a complement of X in Y. Furthermore, let $W' = B^{-1}(X')$. Then W' is a complement of $A(V)$ in W. This yields the decomposition

$$\underset{V}{\bullet} \overset{A}{\hookrightarrow} \underset{W}{\bullet} \overset{B}{\twoheadrightarrow} \underset{Y}{\bullet} \;=\; \underset{V}{\bullet} \xrightarrow{\quad \underset{\sim}{A} \quad} \underset{A(V)}{\bullet} \xrightarrow{\quad \underset{\sim}{B} \quad} \underset{X}{\bullet} \;\oplus\; \underset{0}{\bullet} \xrightarrow{\qquad} \underset{W'}{\bullet} \overset{B}{\twoheadrightarrow} \underset{X'}{\bullet}$$

Here, the first summand is a multiple of $1 \xrightarrow{\sim} 1 \xrightarrow{\sim} 1$. By splitting away the kernel of B, the second summand can be decomposed into multiples of $0 \longrightarrow 1 \xrightarrow{\sim} 1$ and $0 \longrightarrow 1 \longrightarrow 0$. So, on the whole, this quiver has six indecomposable representations:

$$1 \longrightarrow 0 \longrightarrow 0\ , \quad 0 \longrightarrow 0 \longrightarrow 1\ , \quad 1 \xrightarrow{\sim} 1 \longrightarrow 0\ ,$$

$$1 \xrightarrow{\sim} 1 \xrightarrow{\sim} 1\ , \quad 0 \longrightarrow 1 \xrightarrow{\sim} 1\ , \quad 0 \longrightarrow 1 \longrightarrow 0$$

2. Now we look at the orientation

$$\bullet \longrightarrow \bullet \longleftarrow \bullet\ .$$

Very similarly to the other orientation, we can split away objects of the type

$$1 \longrightarrow 0 \longleftarrow 0\ , \quad 0 \longrightarrow 0 \longleftarrow 1$$

which results in a situation where both A and B are injective:

$$\underset{V}{\bullet} \overset{A}{\hookrightarrow} \underset{W}{\bullet} \overset{B}{\hookleftarrow} \underset{Y}{\bullet}\ .$$

By identifying V and Y as subspaces of W, this leads to the problem of classifying pairs of subspaces of a given space W up to isomorphism (the **pairs of subspaces problem**). To do so, we first choose complements V', W', Y' of $V \cap Y$ in V, W, Y, respectively. Then we can decompose the representation as follows:

$$\underset{V}{\bullet} \hookrightarrow \underset{W}{\bullet} \hookleftarrow \underset{Y}{\bullet} \ = \ \underset{V'}{\bullet} \hookrightarrow \underset{W'}{\bullet} \hookleftarrow \underset{Y'}{\bullet} \ \oplus \ \underset{V\cap Y}{\bullet} \xrightarrow{\sim} \underset{V\cap Y}{\bullet} \xleftarrow{\sim} \underset{V\cap Y}{\bullet}.$$

The second summand is a multiple of the object $1 \xrightarrow{\sim} 1 \xleftarrow{\sim} 1$. We go on decomposing the first summand. Again, to simplify notation, we let

$$V = V',\ W = W',\ Y = Y'$$

We can now assume that $V \cap Y = 0$. Next, let W' be a complement of $V \oplus Y$ in W. Then we get

$$\underset{V}{\bullet} \hookrightarrow \underset{W}{\bullet} \hookleftarrow \underset{Y}{\bullet} \ = \ \underset{V}{\bullet} \hookrightarrow \underset{V\oplus Y}{\bullet} \hookleftarrow \underset{Y}{\bullet} \ \oplus \ \underset{0}{\bullet} \longrightarrow \underset{W'}{\bullet} \longleftarrow \underset{0}{\bullet}$$

The second of there summands is a multiple of the indecomposable object $0 \longrightarrow 1 \longleftarrow 0$. The first summand can be further decomposed as follows:

$$\underset{V}{\bullet} \hookrightarrow \underset{V\oplus Y}{\bullet} \hookleftarrow \underset{Y}{\bullet} \ = \ \underset{V}{\bullet} \xrightarrow{\sim} \underset{V}{\bullet} \longleftarrow \underset{0}{\bullet} \ \oplus \ \underset{0}{\bullet} \longrightarrow \underset{Y}{\bullet} \xleftarrow{\sim} \underset{Y}{\bullet}$$

These summands are multiples of

$$1 \xrightarrow{\sim} 1 \longleftarrow 0\ , \quad 0 \longrightarrow 1 \underset{\sim}{\longleftarrow} 1$$

So - like in the other orientation - we get 6 indecomposable representations of A_3:

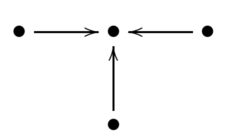
$$1 \longrightarrow 0 \longleftarrow 0\ , \quad 0 \longrightarrow 0 \longleftarrow 1\ , \quad 1 \xrightarrow{\sim} 1 \xleftarrow{\sim} 1\ ,$$

$$0 \longrightarrow 1 \longleftarrow 0\ , \quad 1 \xrightarrow{\sim} 1 \longleftarrow 0\ , \quad 0 \longrightarrow 1 \underset{\sim}{\longleftarrow} 1$$

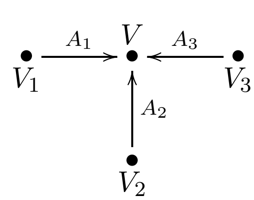
7.3 Indecomposable representations of the quiver D_4

As a last - slightly more complicated - example we consider the quiver D_4.

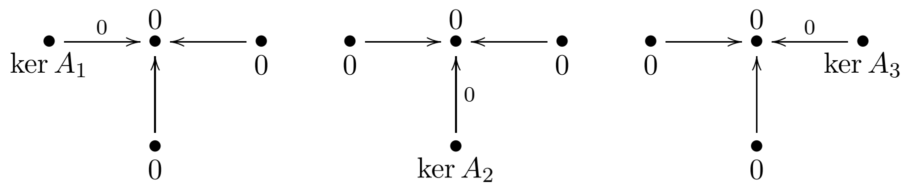
Example 7.9 (D_4)**.** We restrict ourselves to the orientation



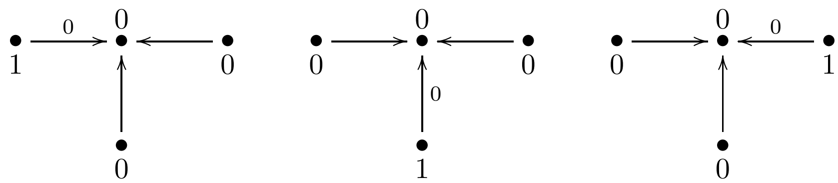
So a representation of this quiver looks like



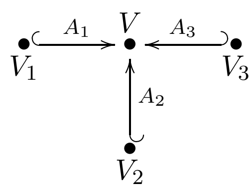
The first thing we can do is - as usual - split away the kernels of the maps A_1, A_2, A_3. More precisely, we split away the representations



These representations are multiples of the indecomposable objects

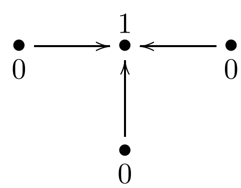


So we get to a situation where all of the maps A_1, A_2, A_3 are injective.



As in 2, we can then identify the spaces V_1, V_2, V_3 with subspaces of V. So we get to the **triple of subspaces problem** of classifying a triple of subspaces of a given space V.

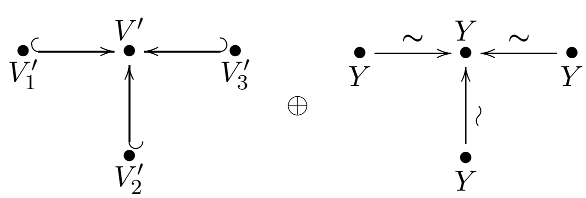
The next step is to split away a multiple of



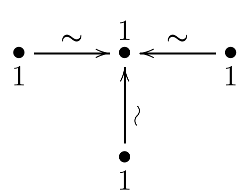
to reach a situation where

$$V_1 + V_2 + V_3 = V.$$

Now, by letting $Y = V_1 \cap V_2 \cap V_3$ and choosing complements V_1', V_2', V_3', V' of Y in V_1, V_2, V_3, V respectively, we can decompose this representation into



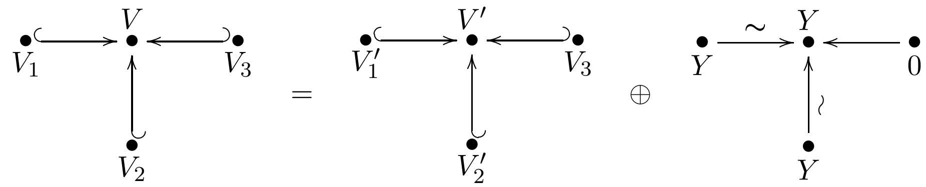
The last summand is a multiple of the indecomposable representation



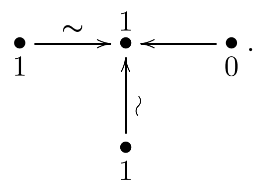
So - considering the first summand and renaming the spaces to simplify notation - we are in a situation where

$$V = V_1 + V_2 + V_3, \quad V_1 \cap V_2 \cap V_3 = 0$$

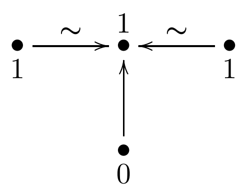
As a next step, we let $Y = V_1 \cap V_2$ and we choose complements V_1', V_2', V' of Y in V_1, V_2 and V, such that $V_3 \subset V'$. This yields the decomposition



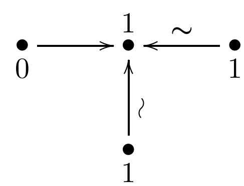
The second summand is a multiple of the indecomposable object



In the resulting situation we have $V_1 \cap V_2 = 0$. Similarly we can split away multiples of



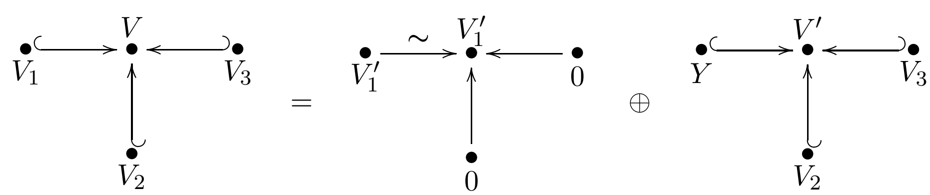
and



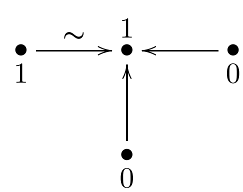
to reach a situation where the spaces V_1, V_2, V_3 do not intersect pairwise

$$V_1 \cap V_2 \; = \; V_1 \cap V_3 \; = \; V_2 \cap V_3 \; = \; 0$$

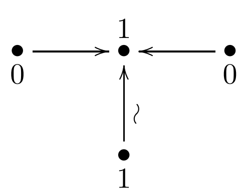
Next, if $V_1 \not\subseteq V_2 \oplus V_3$ we let $Y = V_1 \cap (V_2 \oplus V_3)$. We let V_1' be a complement of Y in V_1. Since then $V_1' \cap (V_2 \oplus V_3) = 0$, we can select a complement V' of V_1' in V which contains $V_2 \oplus V_3$. This gives us the decomposition



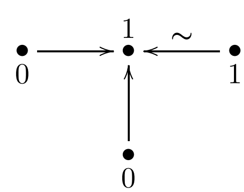
The first of these summands is a multiple of



By splitting these away we get to a situation where $V_1 \subseteq V_2 \oplus V_3$. Similarly, we can split away objects of the type



and



to reach a situation in which the following conditions hold

1. $V_1 + V_2 + V_3 = V$
2. $V_1 \cap V_2 = 0, \quad V_1 \cap V_3 = 0, \quad V_2 \cap V_3 = 0$
3. $V_1 \subseteq V_2 \oplus V_3, \quad V_2 \subseteq V_1 \oplus V_3, \quad V_3 \subseteq V_1 \oplus V_2$

But this implies that

$$V_1 \oplus V_2 = V_1 \oplus V_3 = V_2 \oplus V_3 = V.$$

So we get

$$\dim V_1 = \dim V_2 = \dim V_3 = n$$

and

$$\dim V = 2n$$

Since $V_3 \subseteq V_1 \oplus V_2$ we can write every element of V_3 in the form

$$x \in V_3, \quad x = (x_1, x_2),\ x_1 \in V_1,\ x_2 \in V_2$$

We then can define the projections

$$B_1 : V_3 \to V_1, \quad (x_1, x_2) \mapsto x_1$$

$$B_2 : V_3 \to V_2, \quad (x_1, x_2) \mapsto x_2$$

Since $V_3 \nsubseteq V_1, V_3 \nsubseteq V_2$, these maps have to be injective and therefore are isomorphisms. We then define the isomorphism

$$A = B_2 \circ B_1^{-1}$$

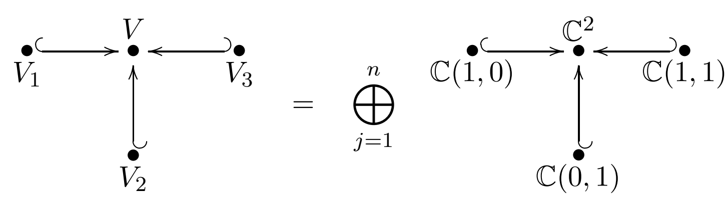
Let $e_1, \ldots, e_n$ be a basis for V_1. Then we get

$$V_1 = \mathbb{C}\, e_1 \oplus \mathbb{C}\, e_2 \oplus \cdots \oplus \mathbb{C}\, e_n$$

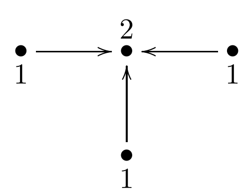
$$V_2 = \mathbb{C}\, Ae_1 \oplus \mathbb{C}\, Ae_2 \oplus \cdots \oplus \mathbb{C}\, Ae_n$$

$$V_3 = \mathbb{C}\,(e_1, Ae_1) \oplus \mathbb{C}\,(e_2, Ae_2) \oplus \cdots \oplus \mathbb{C}\,(e_n, Ae_n)$$

So we can think of V_3 as the graph of an isomorphism $A : V_1 \to V_2$. From this we obtain the decomposition



These correspond to the indecomposable object



Here, the maps coorrespond to the embeddings into the plane of the lines $x = 0$, $y = 0$, and $y = x$.

Thus the quiver D_4 with the selected orientation has 12 indecomposable objects. If one were to explicitly decompose representations for the other possible orientations, one would also find 12 indecomposable objects.

It appears as if the number of indecomposable representations does not depend on the orientation of the edges, and indeed - Gabriel's theorem generalizes this observation.

7.4 Roots

From now on, let Γ be a fixed graph of type A_n, D_n, E_6, E_7, E_8. We denote the adjacency matrix of Γ by R_Γ.

Definition 7.10 (Cartan Matrix)**.** We define the Cartan matrix as

$$A_\Gamma = 2\mathrm{Id} - R_\Gamma$$

On the lattice $\mathbb{Z}^n$ (or the space $\mathbb{R}^n$) we then define an inner product

$$B(x,y) = x^T A_\Gamma y$$

corresponding to the graph Γ.

Lemma 7.11. *1. B is positive definite*

2. $B(x,x)$ takes only even values for $x \in \mathbb{Z}^n$.

Proof. 1. This follows by definition, since Γ is a Dynkin diagram.

2. By the definition of the Cartan matrix we get

$$B(x,x) = x^T Ay = \sum_{i,j} x_i\, a_{ij}\, x_j = 2\sum_i x_i^2 + \sum_{i,j,\, i\neq j} x_i\, a_{ij}\, x_j$$

But since A is symmetric, we obtain

$$B(x,x) = 2\sum_i x_i^2 + \sum_{i,j,\, i\neq j} x_i\, a_{ij}\, x_j = 2\sum_i x_i^2 + 2\cdot \sum_{i<j} a_{ij}\, x_i x_j$$

which is even.

∎

Definition 7.12 (Root)**.** A root is a shortest (with respect to B), nonzero vector in $\mathbb{Z}^n$

So, a root is a nonzero vector $x \in \mathbb{Z}^n$ such that

$$B(x,x) = 2$$

Remark 7.13. There can be only finitely many roots, since all of them have to lie in a ball of some radius.

Definition 7.14. We call vectors of the form

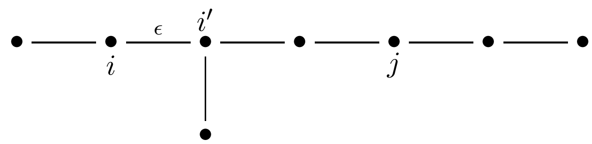
$$\alpha_i = (0, \ldots, \overbrace{1}^{i-\mathrm{th}}, \ldots, 0)$$

simple roots.

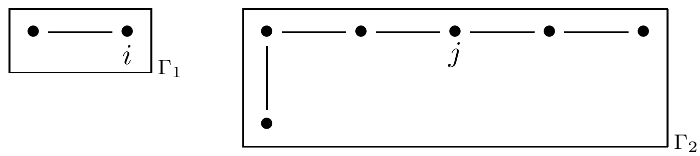
The α_i naturally form a basis of the lattice $\mathbb{Z}^n$.

Lemma 7.15. *Let α be a root, $\alpha = \sum_{i=1}^n k_i\alpha_i$. Then either $k_i \geq 0$ for all i or $k_i \leq 0$ for all i.*

Proof. Assume the contrary, i.e. $k_i > 0$, $k_j < 0$. Without loss of generality, we can also assume that $k_s = 0$ for all s between i and j. We can identify the indices i, j with vertices of the graph Γ.



Next, let ϵ be the edge connecting i with the next vertex towards j and i' be the vertex on the other end of ϵ. We then let Γ_1, Γ_2 be the graphs obtained from Γ by removing ϵ. Since Γ is supposed to be a Dynkin diagram - and therefore has no cycles or loops - both Γ_1 and Γ_2 will be connected graphs, which are not connected to each other.



Then we have $i \in \Gamma_1$, $j \in \Gamma_2$. We define

$$\beta = \sum_{m \in \Gamma_1} k_m \alpha_m, \quad \gamma = \sum_{m \in \Gamma_2} k_m \alpha_m$$

With this choice we get

$$\alpha = \beta + \gamma.$$

Since $k_i > 0, k_j < 0$ we know that $\beta \neq 0, \gamma \neq 0$ and therefore

$$B(\beta, \beta) \geq 2, \quad B(\gamma, \gamma) \geq 2.$$

Furthermore,

$$B(\beta, \gamma) = -k_i k_{i'}$$

since Γ_1, Γ_2 are only connected at ϵ. But this has to be a positive number, since $k_i > 0$ and $k_{i'} \leq 0$ for $i' \in \Gamma_2$. This yields

$$B(\alpha, \alpha) = B(\beta + \gamma, \beta + \gamma) = \underbrace{B(\beta, \beta)}_{\geq 2} + 2\underbrace{B(\beta, \gamma)}_{\geq 0} + \underbrace{B(\gamma, \gamma)}_{\geq 2} \geq 4$$

But this is a contradiction, since α was assumed to be a root. □

Definition 7.16 (positive and negative roots)**.** We call a root $\alpha = \sum_i k_i \alpha_i$ a positive root, if all $k_i \geq 0$. A root which $k_i \leq 0$ for all i is called a negative root.

Remark 7.17. The Lemma states that every root is either positive or negative.

Example 7.18. 1. Let Γ be of the type A_{n-1}. Then the lattice $L = \mathbb{Z}^{n-1}$ can be realized as a subgroup of the lattice $\mathbb{Z}^n$ of all vectors $(x_1, \ldots, x_n)$ such that

$$\sum_i x_i = 0.$$

The vectors

$$\begin{aligned} \alpha_1 &= (1, -1, 0, \ldots, 0) \\ \alpha_2 &= (0, 1, -1, 0, \ldots, 0) \\ &\vdots \\ \alpha_{n-1} &= (0, \ldots, 0, 1, -1) \end{aligned}$$

naturally form a basis of L. Furthermore, the standard inner product

$$(x, y) = \sum x_i y_i$$

on $\mathbb{Z}^n$ restricts to the inner product B given by Γ on L, since it takes the same values on the basis vectors:

$$(\alpha_i, \alpha_i) = 2$$

$$(\alpha_i, \alpha_j) = \begin{cases} -1 & i, j \text{ adjacent} \\ 0 & \text{otherwise} \end{cases}$$

This means that vectors of the form

$$(0, \dots, 0, 1, 0, \dots, 0, -1, 0, \dots, 0) = \alpha_i + \alpha_{i+1} + \cdots + \alpha_{j-1}$$

and

$$(0, \dots, 0, -1, 0, \dots, 0, 1, 0, \dots, 0) = -(\alpha_i + \alpha_{i+1} + \cdots + \alpha_{j-1})$$

are the roots of L. Therefore the number of positive roots in L equals

$$\frac{n(n-1)}{2}$$

2. As a fact we also state the number of positive roots in the other Dynkin diagrams:

D_n	$n(n-1)$
E_6	36 roots
E_7	63 roots
E_8	120 roots

Definition 7.19 (Root reflection)**.** Let $\alpha \in \mathbb{Z}^n$ be a positive root. The reflection s_α is defined by the formula

$$s_\alpha(v) = v - B(v, \alpha)\alpha$$

We denote s_{α_i} by s_i and call these **simple reflections**.

Remark 7.20. s_α fixes B, since

$$B(s_\alpha(v), s_\alpha(w)) = B(v - B(v, \alpha)\alpha, w - B(w, \alpha)\alpha)) =$$

$$B(v, w) - B(v, B(w, \alpha)\alpha) - B(B(v, \alpha)\alpha, w) + B(B(v, \alpha)\alpha, B(w, \alpha)\alpha) = B(v, w)$$

Remark 7.21. As a linear operator of $\mathbb{R}^n$, s_α fixes any vector orthogonal to α and

$$s_\alpha(\alpha) = -\alpha$$

Therefore s_α is the reflection at the hyperplane orthogonal to α. The s_i generate a subgroup $W \subseteq O(\mathbb{R}^n)$. Since for every $w \in W$, $w(\alpha_i)$ is a root, and since there are only finitely many roots, W has to be finite.

7.5 Gabriel's theorem

Definition 7.22. Let Q be a quiver with any labeling $1, \ldots, n$ of the vertices. Let $V = (V_1, \ldots, V_n)$ be a representation of Q. We then call

$$d(V) = (\dim V_1, \ldots, \dim V_n)$$

the dimension vector of this representation.

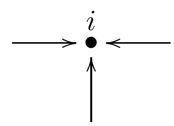
We are now able to formulate Gabriel's theorem using roots.

Theorem 7.23 (Gabriel's theorem)**.** *Let Q be a quiver of type A_n, D_n, E_6, E_7, E_8. Then Q has finitely many indecomposable representations. Namely, the dimension vector of any indecomposable representation is a positive root (with respect to B_Γ) and for any positive root α there is exactly one indecomposable representation with dimension vector α.*

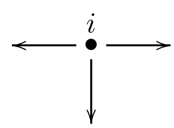
The proof of this theorem is contained in the next three subsections.

7.6 Reflection Functors

Definition 7.24. Let Q be any quiver. We call a vertex $i \in Q$ a sink, if all edges connected to i point towards i.



We call a vertex $i \in Q$ a source, if all edges connected to i point away from i.



Definition 7.25. Let Q be any quiver and $i \in Q$ be a sink (a source). Then we let $\overline{Q_i}$ be the quiver obtained from Q by reversing all arrows pointing into (pointing out of) i.

We are now able to define the reflection functors.

Definition 7.26. Let Q be a quiver, $i \in Q$ be a sink. Let V be a representation of Q. Then we introduce the reflection functor

$$F_i^+ : \mathrm{Rep}Q \to \mathrm{Rep}\overline{Q_i}$$

by the rule

$$F_i^+(V)_k = V_k \quad \text{if } k \neq i$$

$$F_i^+(V)_i = \ker\left(\bigoplus_{j \to i} V_j \to V_i\right)$$

Also, all maps stay the same but those now pointing out of i; these are replaced by the obvious projections.

Definition 7.27. Let Q be a quiver, $i \in Q$ be a source. Let V be a representation of Q. Let ψ be the canonical map

$$\psi : V_i \to \bigoplus_{i \to j} V_j$$

Then we define the reflection functor

$$F_i^- : \mathrm{Rep}Q \to \mathrm{Rep}\overline{Q_i}$$

by the rule

$$F_i^-(V)_k = V_k \quad \text{if } k \neq i$$

$$F_i^-(V)_i = \mathrm{Coker}\,(\psi) = \left(\bigoplus_{i \to j} V_j \right) / (\mathrm{Im}\psi)$$

Again, all maps stay the same but those now pointing into i; these are replaced by the obvious projections.

Proposition 7.28. *Let Q be a quiver, V an indecomposable representation of Q.*

1. *Let $i \in Q$ be a sink. Then either $\dim V_i = 1$, $\dim V_j = 0$ for $j \neq i$* ***or***

$$\varphi : \bigoplus_{j \to i} V_j \to V_i$$

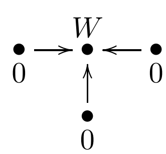
   is surjective.

2. *Let $i \in Q$ be a source. Then either $\dim V_i = 1$, $\dim V_j = 0$ for $j \neq i$* ***or***

$$\psi : V_i \to \bigoplus_{i \to j} V_j$$

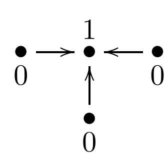
   is injective.

Proof. 1. Choose a complement W of $\mathrm{Im}\varphi$. Then we get

$V \quad =$ 

$\oplus \quad V'$

Since V is indecomposable, one of these summands has to be zero. If the first summand is zero, then φ has to be surjective. If the second summand is zero, then the first has to be of the desired form, because else we could write it as a direct sum of several objects of the type



which is impossible, since V was supposed to be indecomposable.

2. Follows similarly by splitting away the kernel of ψ.

□

Proposition 7.29. *Let Q be a quiver, V be a representation of Q.*

1. *If*

$$\varphi : \bigoplus_{j \to i} V_j \to V_i$$

   is surjective, then

$$F_i^- F_i^+ V = V$$

2. *If*

$$\psi : V_i \to \bigoplus_{i \to j} V_j$$

   is injective, then

$$F_i^+ F_i^- V = V$$

Proof. In the following proof, we will always mean by $i \to j$ that i points into j in the original quiver Q. We only prove the first statement and we also restrict ourselves to showing that the spaces of V and $F_i^- F_i^+ V$ are the same. It is enough to do so for the i-th space. Let

$$\varphi : \bigoplus_{j \to i} V_j \to V_i$$

be surjective and let

$$K = \ker \varphi.$$

When applying F_i^+, the space V_i gets replaced by K. Furthermore, let

$$\psi : K \to \bigoplus_{j \to i} V_j$$

After applying F_i^-, K gets replaced by

$$K' = \left(\bigoplus_{j \to i} V_j \right) / (\mathrm{Im} \psi)$$

But

$$\mathrm{Im} \psi = K$$

and therefore

$$K' = \left(\bigoplus_{j \to i} V_j \right) / \left(\ker \bigoplus_{j \to i} V_j \to V_i \right) = \mathrm{Im} \bigoplus_{j \to i} V_j \to V_i$$

by homomorphism theorem. Since φ was assumed to be surjective, we get

$$K' = V_i$$

□

Proposition 7.30. *Let Q be a quiver, V be an indecomposable representation. Then $F_i^+ V$ and $F_i^- V$ (whenever defined) are either indecomposable or 0*

Proof. We prove the proposition for F_i^+V - the case F_i^-V follows similarly. By 7.28 it follows that either

$$\varphi : \bigoplus_{j\to i} V_j \to V_i$$

is surjective or $\dim V_i = 1, \dim V_j = 0, \quad j \neq i$. In the last case

$$F_i^+V = 0$$

So we can assume that φ is surjective. In this case, assume that F_i^+V is decomposable

$$F_i^+V = X \oplus Y$$

with $X, Y \neq 0$. But F_i^+V is injective at i, since the maps are canonical embeddings. Therefore X and Y also have to be injective at i and hence (by 7.29)

$$F_i^+F_i^-X = X, \quad F_i^+F_i^-Y = Y$$

In particular

$$F_i^-X \neq 0, \quad F_i^-Y \neq 0.$$

Therefore

$$V = F_i^-F_i^+V = F_i^-X \oplus F_i^-Y$$

which is a contradiction, since V was assumed to be indecomposable. So we can infer that

$$F_i^+V$$

is indecomposable. □

Proposition 7.31. *Let Q be a quiver and V a representation of Q.*

1. *Let $i \in Q$ be a sink and let V be surjective at i. Then*
$$d(F_i^+V) = s_i(d(V))$$
2. *Let $i \in Q$ be a source and let V be injective at i. Then*
$$d(F_i^-V) = s_i(d(V))$$

Proof. We only prove the first statement, the second one follows similarly. Let $i \in Q$ be a sink and let

$$\varphi : \bigoplus_{j\to i} V_j \to V_i$$

be surjective. Let $K = \ker \varphi$. Then

$$\dim K = \sum_{j\to i} \dim V_j - \dim V_i$$

Therefore we get

$$\left(d(F_i^+V) - d(V)\right)_i = \sum_{j\to i} \dim V_j - 2\dim V_i = -B\left(d(V), \alpha_i\right)$$

and

$$\left(d(F_i^+V) - d(V)\right)_j = 0, \quad j \neq i.$$

This implies

$$\begin{aligned} & d(F_i^+V) - d(V) = -B\left(d(V), \alpha_i\right)\alpha_i \\ \Leftrightarrow \quad & d(F_i^+V) \ = \ d(V) - B\left(d(V), \alpha_i\right)\alpha_i \ = \ s_i\left(d(V)\right) \end{aligned}$$

□

7.7 Coxeter elements

Definition 7.32. Let Q be a quiver and let Γ be the underlying graph. Fix any labeling $1, \dots, r$ of the vertices of Γ. Then the Coxeter element c of Q corresponding to this labeling is defined as

$$c = s_1 s_2 \dots s_r$$

Lemma 7.33. *Let*

$$\beta = \sum_i k_i \alpha_i$$

with $k_i \geq 0$ for all i but not all $k_i = 0$. Then there is $N \in \mathbb{N}$, such that

$$c^N \beta$$

has at least one strictly negative coefficient.

Proof. c belongs to a finite group W. So there is $M \in \mathbb{N}$, such that

$$c^M = 1$$

We claim that

$$1 + c + c^2 + \cdots + c^{M-1} = 0$$

as operators on $\mathbb{R}^n$. This implies what we need, since β has at least one strictly positive coefficient, so one of the elements

$$c\beta, c^2\beta, \dots, c^{M-1}\beta$$

must have at least one strictly negative one. It is enough to show that 1 is not an eigenvalue for c, since

$$(1 + c + c^2 + \cdots + c^{M-1})v = w \neq 0$$

$$\Rightarrow \quad cw = c\left(1 + c + c^2 + \cdots + c^{M-1}\right)v = (c + c^2 + c^3 + \cdots + c^{M-1} + 1)v = w$$

Assume the contrary, i.e. 1 is a eigenvalue of c and let v be a corresponding eigenvector.

$$cv = v \quad \Rightarrow \quad s_1 \dots s_r v = v$$

$$\Leftrightarrow \quad s_2 \dots s_r v = s_1 v$$

But since s_i only changes the i-th coordinate of v, we get

$$s_1 v = v \quad \text{and} \quad s_2 \dots s_r v = v$$

Repeating the same procedure, we get

$$s_i v = v$$

for all i. But this means

$$B(v, \alpha_i) = 0$$

for all i and since B is nondegenerate, we get $v = 0$. But this is a contradiction, since v is an eigenvector. ☐

7.8 Proof of Gabriel's theorem

Let V be an indecomposable representation of Q. We introduce a fixed labeling $1, \dots r$ on Q, such that $i < j$ if one can reach j from i. This is possible, since we can assign the highest label to any sink, remove this sink from the quiver, assign the next highest label to a sink of the remaining quiver and so on. This way we create a labeling of the desired kind.

We now consider the sequence

$$V^{(0)} = V,\ V^{(1)} = F_r^+ V,\ V^{(2)} = F_{r-1}^+ F_r^+ V, \dots$$

This sequence is well defined because of the selected labeling: r has to be a sink of Q, $r-1$ has to be a sink of $\overline{Q_r}$ and so on. Furthermore we note that $V^{(r)}$ is a representation of Q again, since every arrow has been reversed twice (since we applied a reflection functor to every vertex). This implies that we can define

$$V^{(r+1)} = F_r^+ V^{(r)}, \dots$$

and continue the sequence to infinity.

Theorem 7.34. *There is $m \in \mathbb{N}$, such that*

$$d\left(V^{(m)}\right) = \alpha_p$$

for some p.

Proof. If $V^{(i)}$ is surjective at the appropriate vertex k, then

$$d\left(V^{(i+1)}\right) = d\left(F_k^+ V^{(i)}\right) = s_k d\left(V^{(i)}\right)$$

This implies, that if $V^{(0)}, \dots, V^{(i-1)}$ are surjective at the appropriate vertices, then

$$d\left(V^{(i)}\right) = \dots s_{r-1} s_r d(V)$$

By 7.33 this cannot continue indefinitely - since $d\left(V^{(i)}\right)$ may not have any negative entries. Let i be the smallest number such that $V^{(i)}$ is not surjective at the appropriate vertex. Then $V^{(i)}$ is not surjective at this vertex (which is a sink) but by 7.30 it is indecomposable. So, by 7.28, we get

$$d(V^{(i)}) = \alpha_p$$

for some p. □

Corollary 7.35. *Let Q be a quiver, V be any indecomposable representation. Then $d(V)$ is a positive root.*

Proof. By 7.34

$$s_{i_1} \dots s_{i_n} (d(V)) = \alpha_p.$$

Since the s_i preserve B, we get

$$B(d(V), d(V)) = B(\alpha_p, \alpha_p) = 2.$$

□

Corollary 7.36. *Let V, V' be indecomposable representations of Q such that $d(V) = d(V')$. Then V and V' are isomorphic.*

Proof. Let i be such that

$$d\left(V^{(i)}\right) = \alpha_p.$$

Then we also get $d\left(V'^{(i)}\right) = \alpha_p$. So

$$V'^{(i)} = V^{(i)} =: V^i.$$

Furthermore we have

$$V^{(i)} = F_k^+ \dots F_{r-1}^+ F_r^+ V^{(0)}$$
$$V'^{(i)} = F_k^+ \dots F_{r-1}^+ F_r^+ V'^{(0)}$$

But both $V^{(i-1)}, \dots, V^{(0)}$ and $V'^{(i-1)}, \dots, V'^{(0)}$ have to be surjective at the appropriate vertices. This implies

$$F_r^- F_{r-1}^- \dots F_k^- V^i = \begin{cases} F_r^- F_{r-1}^- \dots F_k^- F_k^+ \dots F_{r-1}^+ F_r^+ V^{(0)} & = V^{(0)} \quad = V \\ F_r^- F_{r-1}^- \dots F_k^- F_k^+ \dots F_{r-1}^+ F_r^+ V'^{(0)} & = V'^{(0)} \quad = V' \end{cases}$$

□

These two corollaries show that there are only finitely many indecomposable representations (since there are only finitely many roots) and that the dimension vector of each of them is a positive root. The last statement of Gabriel's theorem follows from

Corollary 7.37. *For every positive root α, there is an indecomposable representation V with*

$$d(V) = \alpha$$

Proof. Consider the sequence

$$s_r\alpha, s_{r-1}s_r\alpha, \dots$$

Consider the first element of this sequence which is a negative root (this has to happen by 7.33) and look at one step before that, call this element β. So β is a positive root and $s_i\beta$ is a negative root for some i. But since the s_i only change one coordinate, we get

$$\beta = \alpha_i$$

and

$$(s_q \dots s_{r-1}s_r)\alpha = \alpha_i.$$

We let $\mathbb{C}_{(i)}$ be the representation having dimension vector α_i. Then we define

$$V = F_r^- F_{r-1}^- \dots F_q^- \mathbb{C}_{(i)}$$

This is an indecomposable representation and

$$d(V) = \alpha.$$

□