18.705 Fall 2014
Commutative Algebra
Final Exam (Solution)

9:00–12:00

December 16, 2014

E17–122

Your Name	

Problem	Points
1	/40
2	/20
3	/10
4	/10
5	/10
6	/10
7	$/+10$
Total	$/100+10$

Instructions:

1. You may bring and consult the *Lecture Note* and/or your own notes. In principle, you are allowed to view notes in electronic forms. But you are prohibited from consulting anything related to the exam via internet connection.

2. You are allowed to cite any results, even in exercises from the *Lecture Note*, by either referring to the number of the statement or precisely stating it. However, to get full credit, you need to explain how you apply the result you are citing to the specific situation, if not totally obvious.

3. You are encouraged to write down ideas, if not solving completely. However, to write down ideas does not mean to write down everything you know without considering the relevance.

4. Note that: **All rings are commutative and have 1, and ring homomorphisms preserve 1.**

5. Problems start from the next page.

Problem 1. (40 pts, 5 each)

For the following statements, tell whether they are true or false by circling **T** or **F**, respectively. No proofs are required.

1. A finitely generated module over a Noetherian ring is also finitely presented.
   T: By Theorem (16.19).

2. A module has no associated prime if and only if it is the zero module.
   F: One needs the ring to be Noetherian for this to be true.

3. A subring of a Noetherian ring is Noetherian as well.
   F: Any domain that is not Noetherian will give a counter-example since it is a subring of its fraction field, which is always Noetherian.

4. Let R be a Noetherian ring and $\mathfrak{p}$ a prime of R. Then $R_{\mathfrak{p}}$ is Artinian if and only if $\mathfrak{p}$ is a minimal prime.
   T: Since $R_{\mathfrak{p}}$ is Noetherian, it is Artinian if and only if it has dimension 0, which is equivalent to $\mathfrak{p}$ being minimal.

5. Let R be a ring. A filtered direct limit of projective R-modules is flat.
   T: It follows from Proposition (9.19) and the fact that projective modules are flat.

6. Over an Artinian ring, a module has finite length if and only if it is finitely generated.
   T: One direction is always true. For the other one, one uses the condition that the ring is Artinian.

7. If a ring has finite dimension, then it is Noetherian.
   F: There is no relation between being finite dimensional and Noetherian. For example, the ring of the integers in the algebraic closure of $\mathbb{Q}_p$ (an example given in the class) has dimension 1 but is not Noetherian.

8. A module has empty support if and only if it is the zero module.
   T: By Proposition (13.35).

Problem 2. (20 pts, 5 each)

Let k be a field and put $R = k[x_1, x_2]$ where x_i is of degree i for $i = 1, 2$. We may regard R as a graded ring by letting R_n be the k-subspace of R spanned by monomials of degree n for $n \geq 0$ (for example, the monomial $x_1^3 x_2^4$ is of degree $1 \times 3 + 2 \times 4 = 11$). For $n \geq 0$, define $H_R(n)$ to be dimension of R_n over k.

1. Find $H_R(n)$.
2. Show that $H_R(n)$ does not agree with a polynomial function in n, even for $n \gg 0$.
3. Show that $H_R(2n)$ and $H_R(2n+1)$ are both polynomial functions in n for $n \geq 0$.
4. Express the Hilbert series $\sum_{n\geq 0} H_R(n)t^n$ as $\frac{p(t)}{q(t)}$ where p, q are coprime polynomials in $\mathbb{Z}[t]$.

Solution.

1. It is clear that $H_R(n) = \lfloor \frac{n}{2} \rfloor + 1$.
2. Otherwise, the function $H_R(n+1) - H_R(n)$ will agree with a polynomial in n (for $n \gg 0$). It is not a constant but has infinitely many zeros, which is impossible.
3. We have $H_R(2n) = n + 1$, and $H_R(2n+1) = n + 1$.
4. Put $S_0(t) = \sum_{n\geq 0} H_R(2n)t^{2n}$ and $S_1(t) = \sum_{n\geq 0} H_R(2n+1)t^{2n+1}$. Then $S_1(t) = tS_0(t)$. We have

$$\begin{aligned} S_0(t) &= \sum_{n\geq 0} H_R(2n)t^{2n} = \sum_{n\geq 0} (n+1)t^{2n} = \frac{1}{2}\left(\sum_{n\geq 0} t^{2n} + \sum_{n\geq 0}(2n+1)t^{2n}\right) \\ &= \frac{1}{2}\left(\frac{1}{1-t^2} + \left(\frac{t}{1-t^2}\right)'\right) = \frac{1}{(1-t^2)^2}. \end{aligned}$$

Therefore,

$$\sum_{n\geq 0} H_R(n)t^n = S_0(t) + S_1(t) = (1+t)S_0(t) = \frac{1}{(1-t)(1-t^2)}.$$

Problem 3. (10 pts)

Show that if $k = \mathbb{Z}/(2)$, then the ideal $(x, y) \subset k[x, y]/(x, y)^2$ is the union of 3 properly smaller ideals. (This shows how the Prime Avoidance cannot be improved.)

Solution.

There are only 8 elements in the ring $k[x, y]/(x, y)^2$, represented by $\{0, 1, x, y, 1 + x, 1 + y, x + y, 1 + x + y\}$. The ideal (x, y) consists of elements $\{0, x, y, x + y\}$. Put $P_1 = \{0, x\}$, $P_2 = \{0, y\}$ and $P_3 = \{0, x + y\}$. Then they are all properly smaller ideals, and we have $(x, y) = P_1 \cup P_2 \cup P_3$.

Problem 4. (10 pts, 5 each)
Let R be a local ring with the maximal ideal $\mathfrak{m}$.

1. Show that if R is Noetherian, then $\bigcap_{n\geq 1} \mathfrak{m}^n = 0$.

2. Show that if R is Artinian, then $\mathfrak{m}^n = 0$ for some integer $n \geq 1$.

Solution.

1. Put $N = \bigcap_{n\geq 1} \mathfrak{m}^n$. Applying Krull Intersection Theorem (18.29) with $\mathfrak{a} = \mathfrak{m}$ and $M = R$, we have that there exits $a \in \mathfrak{m}$ such that $(1 + a)N = 0$. Since R is local, by Proposition (3.2), the element $1 + a$ is invertible. Thus $N = 0$.

2. Since R is Artinian, it is Noetherian, and by DCC we have $\mathfrak{m}^n = \mathfrak{m}^{n+1} = \mathfrak{m}^{n+2} = \cdots$ for n large. Thus $\mathfrak{m}^n = 0$ for some integer $n \geq 1$ by Part 1.

Problem 5. (10 pts)

Let R be a ring and let S be an R-algebra that is separated and complete with respect to an ideal $\mathfrak{n}$ of S. Given $f_1, \ldots, f_n \in \mathfrak{n}$, show that there is a unique R-algebra homomorphism

$$\varphi \colon R[[x_1, \ldots, x_n]] \to S$$

sending x_i to f_i $(i = 1, \ldots, n)$ and taking Cauchy sequences to Cauchy sequences. Here, we recall that $R[[x_1, \ldots, x_n]]$ is the power series R-algebra of n variables. (You get 5 pts if you can show the uniqueness of φ.)

Solution.

First we show that φ is unique if exists. In fact, the restriction of φ to the subalgebra $R[x_1, \ldots, x_n] \subset R[[x_1, \ldots, x_n]]$ is unique as $\varphi(x_i) = f_i$. As $R[[x_1, \ldots, x_n]]$ is isomorphic to the completion of $R[x_1, \ldots, x_n]$ with respect to the ideal $(x_1, \ldots, x_n)$, φ must be unique since it sends Cauchy sequences to Cauchy sequences.

Now we prove the existence of φ. Since S is separated and complete with respect to the ideal $\mathfrak{n}$. We have the isomorphism $S \simeq \varprojlim_{r \geq 1} S/\mathfrak{n}^r$. For $r \geq 1$, let $\varphi_r \colon R[x_1, \ldots, x_n] \to S/\mathfrak{n}^r$ be the unique homomorphism such that $\varphi_r(x_i) = f_i$ $(i = 1, \ldots, n)$. Then φ_r factors through the quotient $R[x_1, \ldots, x_n]/(x_1, \ldots, x_n)^r$, which is isomorphic to $R[[x_1, \ldots, x_n]]/(x_1, \ldots, x_n)^r$. Therefore, φ_r may be viewed as a homomorphism from $R[[x_1, \ldots, x_n]]$ to $S/\mathfrak{n}^r$. From the construction, we have $\varphi_r = \pi_r \circ \varphi_{r+1}$ where $\pi_r \colon S/\mathfrak{n}^{r+1} \to S/\mathfrak{n}^r$ is the projection. By the universal mapping property of projective limits, we obtain a homomorphism $\varphi \colon R[[x_1, \ldots, x_n]] \to S$. It is clear that φ satisfies the requirements.

Problem 6. (10 pts, 5 each)
Let R be a ring and let M, N be two finitely generated R-modules such that $M \otimes_R N = 0$.

1. Show that if R is local, then either M or N is 0.
2. Given an example of R, M and N such that neither M nor N is 0.

Solution.

1. Since both M and N are finitely generated, by Proposition (13.30) we have $\mathrm{Supp}(M) \cap \mathrm{Supp}(N) = \emptyset$. Let $\mathfrak{m}$ be the unique maximal ideal of R. Then without lost of generality, we may assume $\mathfrak{m} \notin \mathrm{Supp}(M)$, that is, $M_{\mathfrak{m}} = 0$. Then $M = 0$ by Proposition (13.35).
2. Take $R = \mathbb{Z}$, $M = \mathbb{Z}/(p)$ and $N = \mathbb{Z}/(q)$, where p, q are two different primes.

Problem 7. (extra credit: 10 pts, 5 each)
Let p be a prime. Put

$$\mathbb{Z}_p\langle T\rangle = \left\{ \sum_{n=0}^{\infty} a_n T^n \mid a_n \in \mathbb{Z}_p, \mathrm{ord}(a_n) \to +\infty \text{ when } n \to +\infty \right\},$$

where for $a \in \mathbb{Z}_p$, $\mathrm{ord}(a) = \sup\{m \,|\, a \in p^m\mathbb{Z}_p\}$ (for example, $\mathrm{ord}(p^2) = 2$ and $\mathrm{ord}(0) = +\infty$). In particular, we have $\mathbb{Z}[T] \subset \mathbb{Z}_p\langle T\rangle \subset \mathbb{Z}_p[[T]]$ as sets. Show that

1. $\mathbb{Z}_p\langle T\rangle$ is a subring of $\mathbb{Z}_p[[T]]$;
2. $\mathbb{Z}_p\langle T\rangle$ is the completion of the subring $\mathbb{Z}[T]$ with respect to the ideal $(p) \subset \mathbb{Z}[T]$.

Solution.

1. We note that for $a, b \in \mathbb{Z}_p$, $\mathrm{ord}(a) \geq 0$; $\mathrm{ord}(ab) = \mathrm{ord}(a) + \mathrm{ord}(b)$; and $\mathrm{ord}(a+b) \geq \min\{\mathrm{ord}(a), \mathrm{ord}(b)\}$. The only nontrivial part of showing $\mathbb{Z}_p\langle T\rangle$ is a subring of $\mathbb{Z}_p[[T]]$ is that $\mathbb{Z}_p\langle T\rangle$ is closed under multiplication. For this, we pick up $f = \sum_{n=0}^{\infty} a_n T^n$ and $g = \sum_{n=0}^{\infty} b_n T^n$ in $\mathbb{Z}_p\langle T\rangle$, and suppose $h = fg = \sum_{n=0}^{\infty} c_n T^n$. We have

$$\mathrm{ord}(c_n) \geq \min\left\{\mathrm{ord}(a_{\lfloor n/2\rfloor}), \cdots, \mathrm{ord}(a_n), \mathrm{ord}(b_{\lfloor n/2\rfloor}), \cdots, \mathrm{ord}(b_n)\right\}.$$

Then an easy limit argument shows that $\mathrm{ord}(c_n) \to +\infty$ when $n \to +\infty$.

2. It is easy to see that for every fixed $r \geq 1$, the natural homomorphism

$$\mathbb{Z}[T]/(p)^r \to \mathbb{Z}_p\langle T\rangle/(p)^r$$

is an isomorphism. Thus we have isomorphisms

$$\widehat{\mathbb{Z}[T]} \simeq \varprojlim_{r\geq 1} \mathbb{Z}[T]/(p)^r \simeq \varprojlim_{r\geq 1} \mathbb{Z}_p\langle T\rangle/(p)^r.$$

The last step is to show that the natural homomorphism $\mathbb{Z}_p\langle T\rangle \to \varprojlim_{r\geq 1} \mathbb{Z}_p\langle T\rangle/(p)^r$ is an isomorphism, or equivalently, $\mathbb{Z}_p\langle T\rangle$ is separated and complete with respect to the ideal (p). We clearly have $\bigcap_{r\geq 1}(p)^r = 0$. Now suppose $\{f_i = \sum_{n=0}^{\infty} a_{i,n}T^n\}$ is a Cauchy sequence in $\mathbb{Z}_p\langle T\rangle$. Then for each n, $\{a_{i,n}\}$ is a Cauchy sequence in $\mathbb{Z}_p$ hence admits a limit $b_n \in \mathbb{Z}_p$. We only need to show that $g := \sum_{n=0}^{\infty} b_n T^n$ belongs to $\mathbb{Z}_p\langle T\rangle$. In fact, for any $C \geq 0$, there is an integer M_C such that $\mathrm{ord}(a_{m+1,n} - a_{m,n}) \geq C$ for every $m \geq M_C$ and every $n \geq 0$. Let N_C be an integer such that $\mathrm{ord}(a_{M_C,n}) \geq C$ for every $n \geq N_C$. Then for every $m \geq M_C$ and $n \geq N_C$, $\mathrm{ord}(a_{m,n}) \geq C$. In particular, for $n \geq N_C$, $\mathrm{ord}(b_n) \geq C$, which completes the proof.