## Lecture 9.

Let us prove Kashiwara's theorem. We need first to show that the image of $i_{*0}$ is contained in $\mathcal{M}_X^r(\mathcal{D}_Y)$. By definition, $i_{*0}(M) = M \otimes_{\mathcal{D}_X} \mathcal{D}_{X\to Y}$, and for a closed embedding $i: X \to Y$, $\mathcal{D}_{X\to Y} = \mathcal{D}_Y / \mathcal{J}\mathcal{D}_Y$, where $\mathcal{J}$ is the ideal of $X$. Every element of $\mathcal{D}_{X\to Y}$ is killed by a large power of $\mathcal{J}$, i.e. for every $L \in \mathcal{D}_Y$ $\exists N$ such that $L\mathcal{J}^N \subset \mathcal{J}\mathcal{D}_Y$; namely, we can take $N = \mathrm{ord}(L)+1$. (It's enough to show this for $L = v_1 \cdots v_r$, where $v_1,..,v_r$ are vector fields). This means that $i_{*0}(M)$ is set-theoretically supported on $X$, i.e. lies in $\mathcal{M}_X(\mathcal{D}_Y)$.

To prove Kashiwara's theorem, we will construct a functor $i^{!0}: \mathcal{M}^r(\mathcal{D}_Y) \to \mathcal{M}^r(\mathcal{D}_X)$, which will be the inverse of $i_{*0}$ when restricted to $\mathcal{M}_X(\mathcal{D}_Y)$. For every $M \in \mathcal{M}(\mathcal{D}_Y)$, define $i^{!0}M = \mathrm{Hom}_{\mathcal{O}_Y}(\mathcal{O}_X, M) \cong$ $\cong \{m \in M \mid m\mathcal{J} = 0\}$. The structure of a $\mathcal{D}_X$-module on this space is given as follows: Lemma 1. Any vector field $v \in \mathrm{Vec}_X$ can ~~always~~ be extended to a vector field $\tilde{v}$ on $Y$ which preserves $\mathcal{J}$.

*Proof.* We need to show that the restriction map $\Gamma(Y, TY) \to \Gamma(X, TY|_X)$ is surjective.

But in fact, if $E$ is any vector bundle on $Y$, then the restriction map $\Gamma(Y, E) \to \Gamma(X, E|_X)$ is surjective, with kernel $\mathcal{J}\Gamma(Y, E)$ (this can be easily checked on formal neighborhoods of points of $X$).

Now we define the action of $v$ on $i^! M$ by $mv = m\tilde{v}$ $\forall m \in M$. This definition does not depend on the choice of $\tilde{v}$. Indeed, if $\tilde{\tilde{v}}$ is another lift then $\tilde{v} - \tilde{\tilde{v}} \in \mathcal{J}.\mathrm{Vect}X$, so $m(\tilde{v} - \tilde{\tilde{v}}) = 0$.

We will in fact prove a stronger version of Kashiwara's theorem.

**Theorem 2.** 1) $i^{!0}$ is right adjoint to $i_{*0}$.

2) The functors $\mathcal{M}^r(\mathcal{D}_X) \underset{i^{*!}}{\overset{i_{*0}}{\rightleftarrows}} \mathcal{M}^r_X(\mathcal{D}_Y)$ are mutually inverse.

*Proof.* To prove 1), we want to show that $\mathrm{Hom}(i_{*0} M, N) \cong \mathrm{Hom}(M, i^{!0} N)$.

There exists a linear map $M \hookrightarrow i_{*0} M = M \otimes_{\mathcal{D}_X} \mathcal{D}_{X \to Y}$ given by $m \mapsto m \otimes 1$

Given $f \in Hom(i_{*0} M, N)$, let's consider the restriction of $f$ to $M$. It is easy to see that this map in fact lands in $i^{!0} N \subset N$. Conversely, given $g \in Hom(M, i^{!0}N)$, we want to construct $\tilde{g} : M \otimes_{\mathcal{D}_X} \mathcal{D}_{X \to Y} \to N$. We know that $\mathcal{D}_{X\to Y} = \mathcal{D}_Y / \mathcal{J}\mathcal{D}_Y$. So a map
$m \otimes L \mapsto g(m) L$ is well defined, since $g(m)$ is killed by $\mathcal{J}$. It's easy to check that the assignments we have constructed are mutually inverse, so the first part of the theorem is proved.

Since $i_{*0}$ and $i^{!0}$ are adjoint, we have canonical adjunction morphisms
$i_{*0} i^{!0} N \to N$, $N \in \mathcal{M}_X(\mathcal{D}_Y)$, and $M \to i^{!0} i_{*0} M$, $M \in \mathcal{M}(\mathcal{D}_X)$. To prove the second part of the theorem, we have to show that they are isomorphisms. It's enough to show this locally. By induction on codimension it's enough to assume that $X$ is a smooth hypersurface in $Y$, given by an equation $f = 0$. Locally we can choose a coordinate system

$y_1, \dots, y_m, \partial_1, \dots, \partial_m$ on $Y$ such that $y_m = f$. We claim that $\mathcal{D}_{X\to Y}$ is a free module over $\mathcal{D}_X$: $\mathcal{D}_{X\to Y} = \bigoplus_n \mathcal{D}_X \partial^n$ where $\partial = \partial_m$ is the vector field satisfying $\partial(f) = 1$. Indeed, $\mathcal{D}_{X\to Y} = \mathcal{O}(X) \otimes_{\mathcal{O}(Y)} \mathcal{D}_Y = \bigoplus \mathcal{O}_X \partial_1^{\alpha_1} \dots \partial_m^{\alpha_m}$, and $\bigoplus_{\alpha_1, \dots, \alpha_{m-1}} \mathcal{O}_X \partial_1^{\alpha_1} \dots \partial_{m-1}^{\alpha_{m-1}}$ is $\mathcal{D}_X$.

Let's prove that $Id \to i^! i_{*}^{0}$ is an isomorphism, i.e. $M \longrightarrow i^! i_*^0 M$ is an isomorphism for any $M \in \mathcal{D}_X$. We have

$$i_*^0 M = M \otimes_{\mathcal{D}_X} \mathcal{D}_{X\to Y} = M \otimes_{\mathcal{D}_X} \left( \bigoplus_j \mathcal{D}_X \partial^j \right) = \bigoplus_j M \partial^j$$

We have a map $f: M\partial^j \to M\partial^{j-1}$ (since $f\partial$ acts on $M\partial^j$ by $j$, as can be proved by induction). So $\ker f|_{i_*^0 M} = M$, hence $i^! i_*^0 M = M$.

Now we have to show that $i_*^0 i^! N \to N$ is an isomorphism. Let $N$ be a $\mathcal{D}_Y$-module supported set-theoretically on $X$, and $\ker f = S \subset N$. By definition $S = i^! N$. We'll prove that $N = i_*^0 S$, i.e. $N = \bigoplus_i S \partial^i$.

Consider $\tilde{N} = \sum_j S\partial^j \subset N$ and $E = f\partial$. On $S\partial^j$, $E$ acts with eigenvalue $j$. This can be proved by induction: if $nE = \lambda n$ then $n\partial E = n(E+1)\partial = (\lambda+1)n\partial$.

So $\tilde{N}$ is in fact a direct sum, and $f: S\partial^j \to S\partial^{j-1}$ is surjective on $\tilde{N}$. ($\forall v \in S\partial^{j-1}$ $v = \frac{v\partial f}{j}$ as $\partial f = f\partial + 1$)

Consider $Q = N/\tilde{N}$. To show that $Q = 0$, it's enough to show that $\operatorname{Ker} f|_Q = 0$ (since we know $f$ acts locally nilpotently on $Q$). Let $z \in N$ such that $zf \in \tilde{N}$. We want to show that in fact $z \in \tilde{N}$. Since $f$ is surjective on $\tilde{N}$, there exists $\tilde{z} \in \tilde{N}$ such that $zf = \tilde{z}f$. So $(z - \tilde{z})f = 0$, hence $z - \tilde{z} \in S \subset \tilde{N}$, hence $z \in \tilde{N}$, as desired. Kashiwara's thm is proved ∎.

One of the applications of Kashiwara's theorem is that you can use it to define $\mathcal{D}$-modules on singular varieties. Namely, let $X$ be any affine variety (possibly singular). Then there exists a closed embedding $i: X \to \mathbb{A}^N$, Then we define $\mathcal{M}(\mathcal{D}_X)=\mathcal{M}^i(\mathcal{D}_X)$ to be the category $\mathcal{M}_X^i(\mathcal{D}_{\mathbb{A}^n})$. By Kashiwara's theorem, this definition coincides with the old one for smooth varieties. We claim that this definition does not depend on the embedding $i: X \hookrightarrow \mathbb{A}^n$. Indeed, let $i_1: X \to \mathbb{A}^{n_1}$, $i_2: X \to \mathbb{A}^{n_2}$ be two embeddings, and let $\mathcal{M}_{i_1}(\mathcal{D}_X)$, $\mathcal{M}_{i_2}(\mathcal{D}_X)$ be the correspon-ding categories. Then we'll construct a canonical equivalence $\mathcal{E}_{i_2 i_1}: \mathcal{M}_{i_1}(\mathcal{D}_X) \to \mathcal{M}_{i_2}^{(}$

First consider the case when $X \overset{i_1}{\hookrightarrow} \mathbb{A}^{n_1} \overset{\pi}{\hookrightarrow} \mathbb{A}^{n_2}$, and $i_2 = \pi \circ i_1$, where $\pi$ is a closed embedding. Then the equivalence $\mathcal{E}_{i_2 i_1}: \mathcal{M}_{i_1}(\mathcal{D}_X) \to \mathcal{M}_{i_2}(\mathcal{D}_X)$ is given by the functor $\pi_{*0}: \mathcal{M}_X(\mathcal{D}_{\mathbb{A}^{n_1}}) \to \mathcal{M}_X(\mathcal{D}_{\mathbb{A}^{n_2}})$ (it's easy to see that $\pi_{*0}$ defines an equivalence between these categories - it follows from Kashiwara's theorem).

Now consider the general case, i.e. we ha

$$\begin{tikzcd} & \mathbb{A}^{n_2} \\ X \arrow[ur, "i_2"] \arrow[r, hook, "i_1"'] & \mathbb{A}^{n_1} \arrow[u, dashed, "\theta"'] \end{tikzcd}$$

We can choose $\theta$ so that this diagram is commutative (since $\mathcal{O}(\mathbb{A}^{n_2})$ is free).

Now consider the diagram

$$\begin{tikzcd} & A^{n_2} & A^{n_1+n_2} \\ X \arrow[ur, hook, "i_2"] \arrow[r, hook, "i_1"'] & A^{n_1} \arrow[u, "\theta"] \arrow[ur, "(\theta,1)"'] & \end{tikzcd}$$

The composition $(\theta,1)\circ i_1 = (i_1, i_2): X \hookrightarrow A^{n_1+n_2}$ is the diagonal map. So we can define

$\varepsilon_{(i_1,i_2),i_1} = (\theta,1)_{*0}$. It is easy to check that it's independent of $\theta$. Now we define $\varepsilon_{i_2 i_1} = \varepsilon^{-1}_{(i_1,i_2),i_2}\,\varepsilon_{(i_1,i_2),i_1}$.

To prove that $M(D_X)$ is canonically defined Thus $\varepsilon_{i_1,i_2} = \varepsilon^{-1}_{i_2,i_1}$,

It remains to show that

$\varepsilon_{i_3 i_2}\circ\varepsilon_{i_2 i_1} = \varepsilon_{i_3 i_1}$, i.e.

$\cancel{\varepsilon^{-1}_{(i_2,i_3),i_3}\,\varepsilon_{(i_2,i_3),i_2}\,\varepsilon^{-1}_{(i_1,i_2),i_2}\,\varepsilon_{(i_1,i_2),i_1}}$

$= \cancel{\varepsilon^{-1}_{(i_1,i_3),i_3}\,\varepsilon_{(i_1,i_3),i_1}}$

$\varepsilon_{i_3,(i_3,i_2)}\,\varepsilon_{(i_3,i_2),i_2}\,\varepsilon_{i_2,(i_2,i_1)}\,\varepsilon_{(i_2,i_1),i_1} =$

$= \varepsilon_{i_3,(i_3,i_1)}\,\varepsilon_{(i_3,i_1),i_1}$.

We have

$RHS = \varepsilon_{i_3,(i_3,i_2,i_1)}\,\varepsilon_{(i_3,i_2,i_1),(i_3,i_1)}\,\varepsilon_{(i_3,i_1),i_1}$

$= \varepsilon_{i_3\,(i_3,i_2,i_1)}\,\varepsilon_{(i_3,i_2,i_1),i_1}$,

<u>Remark.</u>
More precisely, we need to define an isomorphism of functors $\phi_{i_3,i_2,i_1} : \mathcal{E}_{i_3 i_2} \circ \mathcal{E}_{i_2 i_1} \xrightarrow{\sim} \mathcal{E}_{i_3 i_1}$, and show that $\phi_{i_4,i_3,i_1} \circ \phi_{i_3,i_2,i_1} =$

$= \phi_{i_4,i_2,i_1} \circ \phi_{i_4,i_3,i_2}$ (cocycle condition), which is what we are really doing.

$$LHS = \varepsilon_{i_3,(i_3,i_2,i_1)}\,\varepsilon_{(i_3,i_2,i_1),i_2}\,\varepsilon_{i_2,(i_3,i_2,i_1)}\,\varepsilon_{(i_3,i_2,i_1),}$$

$$= \varepsilon_{i_3,(i_3,i_2,i_1)}\,\varepsilon_{(i_3,i_2,i_1),i_1}, \text{ as desired.}$$

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Note that we have a functor $\Gamma: \mathcal{M}^r_X(\mathcal{D}_{\mathbb{A}^n}) \to \mathcal{M}(\mathcal{O}_X)$ given by $M \mapsto \{m \in M, mJ = 0\}$. It's easy to check that if we have two embeddings $i_1: X \to \mathbb{A}^{n_1}$, $i_2: X \to \mathbb{A}^{n_2}$ then the diagram

$$\begin{tikzcd}
\mathcal{M}^r_X(\mathcal{D}_{\mathbb{A}^{n_1}}) \arrow[d, "\varepsilon_{i_2 i_1}"] \arrow[dr, "\Gamma"] & \\
\mathcal{M}_X(\mathcal{D}_{\mathbb{A}^{n_2}}) \arrow[r, "\Gamma"'] & \mathcal{M}(\mathcal{O}_X)
\end{tikzcd}$$

commutes. So we have a functor $\Gamma: \mathcal{M}(\mathcal{D}_X) \to \mathcal{M}(\mathcal{O}_X)$ for an arbitrary affine variety. It's called the functor of global sections (since for smooth $X$ $\Gamma(M) = M = \Gamma(X, M)$).

<u>Remark.</u> Note that if $Y$ is any smooth variety then we have a canonical equivalence $i: X \hookrightarrow Y$, $i\mathcal{M}(\mathcal{D}_X) \to \mathcal{M}_X(\mathcal{D}_Y)$. Indeed, ~~if X → A^n we pick a lift~~ we can see it by embedding $Y$ into affine space ~~and get~~ and using Kashiwara's theorem.

Remark. The functor $\Gamma$ is representable. Namely, let $\tilde{\mathcal{D}}_X \in \mathcal{M}_X(\mathcal{D}_Y)$ be the $\mathcal{D}$-module $\mathcal{D}_Y / \mathcal{J}\mathcal{D}_Y$, where $\mathcal{J}$ is the ideal cutting out $X \subset Y$. It's easy to check that if $i_1 : X \hookrightarrow Y_1$, $i_2 : X \hookrightarrow Y_2$ are two embeddings, and $\varepsilon_{i_2 i_1}$ the corr. equivalence then $\varepsilon_{i_2 i_1}(\tilde{\mathcal{D}}_{X, i_1}) = \tilde{\mathcal{D}}_{X, i_2}$. So $\tilde{\mathcal{D}}_X \in \mathcal{M}(\mathcal{D}_X)$ is a canonically defined object.

Note that if $X$ is singular, $\mathcal{D}_X$ is not an algebra, and does not coincide with the algebra of Grothendieck differential operators $\mathcal{D}(X)$. However, one can show that $\mathcal{D}(X) = \mathrm{End}(\mathcal{D}_X)$. Since $\Gamma(M) = \mathrm{Hom}(\mathcal{D}_X, M)$, $\mathcal{D}(X)$ acts on the space $\Gamma(M)$, although this action does not allow us to reconstruct $M$. Also not that if $X$ is singular, $\mathcal{D}_X$ may not be projective, and correspondingly the functor $\Gamma$ may not be exact (but it is of course exact in the smooth case)

~~Remark. Similarly we can define [illegible] for singular X. One can check this is independent on the [illegible].~~

Appendix 2 to Lecture 9.

Suppose that $X$ is an affine scheme of finite type, and $\bar{X} \subset X$ the corresponding variety (reduced scheme). I.e., $\mathcal{O}(\bar{X}) = \mathcal{O}(X)/_J$, where $J$ is the nilradi[cal]. Then the category $\mathcal{M}(\mathcal{D}_X) = \mathcal{M}^r(\mathcal{D}_X)$ is defined in the same way as for varieties, i.e. $\mathcal{M}(\mathcal{D}_X) = \mathcal{M}_X(\mathcal{D}_Y)$ for an closed embedding $i : X \to Y$ of $X$ into a smooth variety $Y$ (e.g. affine space). Clearly, $\mathcal{M}_X(\mathcal{D}_Y) = \mathcal{M}_{\bar{X}}(\mathcal{D}_Y)$, so $\mathcal{M}(\mathcal{D}_X) = \mathcal{M}(\mathcal{D}_{\bar{X}})$. However, $\Gamma(X, ?) \neq \Gamma(\bar{X}, ?)$ on this category, and $\mathcal{D}_X \neq \mathcal{D}_{\bar{X}}$ (as $\mathcal{D}_X = \mathcal{D}_Y/_{I\mathcal{D}_Y}$, $I$ being the ideal of $X$, and $\mathcal{D}_{\bar{X}} = \mathcal{D}_Y/_{\sqrt{I}\cdot\mathcal{D}_Y}$).

This relates to the notion of !-crystal. Namely, let $M$ be a right $\mathcal{D}_X$-module ($X$-variety). Then $M$ defines an object $M$ in every category $\mathcal{M}(\mathcal{D}_{\tilde{X}})$, where $\tilde{X}$ is any affine scheme of finite type such that $\bar{\tilde{X}} = X$. So we can define the $\mathcal{O}_{\tilde{X}}$-module $M_{\tilde{X}} = \Gamma(\tilde{X}, M)$. These modules are equipped with isomorphisms $\alpha_\eta : M_{\tilde{X}} \to \eta^! M_{\tilde{\tilde{X}}}$ for any finite map of extensions $\eta : \tilde{X} \to \tilde{\tilde{X}}$ (finite map of schemes that is $= id$ on $X$), which are compatible with compositions. Such a structure is called a !-crystal on $X$.

<u>Theorem</u>. The category $Crys(X)$ of !-crystals on $X$ is equivalent by the above functor to $\mathcal{M}^r(\mathcal{D}_X)$.

## Appendix 2 to lecture 9.

Having defined $\mathcal{M}(\mathcal{D}_X)$ for singular $X$, we should define the functor $\pi_{*0}$ ~~[illegible]~~ when $\pi: X \to Y$, and $X, Y$ are possibly singular. To define $\pi_{*0}: \mathcal{M}^2(\mathcal{D}_X) \to \mathcal{M}^2(\mathcal{D}_Y) = \mathcal{M}^2_Y(\mathbb{A}^n)$ (for some $i: Y \hookrightarrow \mathbb{A}^n$), it suffices to define it in the case $Y = \mathbb{A}^n$. So assume $\pi: X \to \mathbb{A}^n$, then for any $i': X \hookrightarrow \mathbb{A}^m$ we have a commutative diagram

$$\begin{tikzcd} & \mathbb{A}^m \arrow[d, "\theta"] \\ X \arrow[ur, "i'"] \arrow[r, "\pi"] & \mathbb{A}^n \end{tikzcd}$$

such that $\theta \circ i' = \pi$.

So we ~~really~~ can ~~need to~~ define $\pi_{*0} = \theta_{*0} \circ i'_{*0}$, where $i'_{*0}$ is the natural equivalence $\mathcal{M}^2(\mathcal{D}_X) \to \mathcal{M}^2_X(\mathcal{D}(\mathbb{A}^m))$.

~~Now let's define $\pi_{*0}: \mathcal{M}^e_Y(\mathcal{D}(\mathbb{A}^m)) \to \mathcal{M}^e(\mathcal{D}_X) = \mathcal{M}^e_X(\mathcal{D}(\mathbb{A}^m))$ for some $i': X \hookrightarrow \mathbb{A}^m$, $i: Y \hookrightarrow \mathbb{A}^n$.~~