Lecture 8.

In more detail: (proof of lemma 9 from last lecture).

$$\omega\cdot fv = -L_{fv}\omega = -d\, i_{fv}\omega = -d(f i_v\omega) =$$
$$= -df\wedge i_v\omega - f\, d i_v\omega = -L_v f\cdot\omega - f L_v\omega =$$
$$= \text{[illegible]}\; -L_v(f\omega) = \omega f\cdot v$$

Lemma 1. Let M be a left $\mathcal{D}_X$-module. Then $M\otimes_{\mathcal{O}(X)}\Omega^n(X)$ is a right $\mathcal{D}_X$-module, with

$$(m\otimes\omega)v = -vm\otimes\omega - m\otimes L_v\omega$$

Proof. First show it's well defined. For this we need to show that

$$(fm\otimes\omega - m\otimes f\omega)v = 0.$$

We have

$$(fm\otimes\omega - m\otimes f\omega)v = -v(fm)\otimes\omega + vm\otimes f\omega$$
$$- fm\otimes L_v\omega + m\otimes L_v(f\omega)$$
$$= -L_v f\cdot m\otimes\omega - f\,vm\otimes\omega + vm\otimes f\omega$$
$$- fm\otimes L_v\omega + m\otimes fL_v\omega + m\otimes L_v f\cdot\omega = 0.$$

Also clearly $(m\otimes\omega)[v,w] = (m\otimes\omega)\cdot v\cdot w - (m\otimes\omega)\cdot w\cdot v$, and

$(m\otimes\omega)\cdot fv = f(m\otimes\omega)\cdot v$ (as follows from the previous lemma). ∎

<u>Remark</u>. Similarly, if M is a right $\mathcal{D}_X$-module then $M \otimes_{\mathcal{O}(X)} \Omega^n(X)^* = M \otimes_{\mathcal{O}(X)} \Lambda^n TX$ is a left $\mathcal{D}_X$-module.

<u>Corollary 2</u> The categories $\mathcal{M}^l(\mathcal{D}_X)$ and $\mathcal{M}^r(\mathcal{D}_X)$ are canonically equivalent. The equivalence is given by $M \mapsto M \otimes_{\mathcal{O}(X)} \Omega^n(X)$.

So we will not make a distinction between these categories and call the category of $\mathcal{D}_X$-modules $\mathcal{M}(\mathcal{D}_X)$.

<u>Remark</u>. The algebras $\mathcal{D}_X$ and $\mathcal{D}_X^{op}$ are not necessarily isomorphic, but they are canonically Morita equivalent.

Let M be a finitely generated $\mathcal{D}_X$-module. Then it has a good filtration F for the "geometric" filtration on $\mathcal{D}_X$ (which is the only one available for general X) and $\mathrm{supp}(\mathrm{gr}^F M) = SS(M)$ does not depend on the filtration. $SS(M)$ is a subvariety of T^*X called the singular support of M. By Gabber's theorem it's ~~independent~~ coisotropic, so $\dim SS(M) \geq n$.

If $\dim SS(M) = n$ ($\Leftrightarrow SS(M)$ is Lagrangian), then we say that M is holonomic.

Assume M is holonomic. Then we can define a cycle (the singular cycle of M) in the following way: let $Z_1, \dots, Z_k$ be the irreducible components of $\mathrm{supp}(\mathrm{gr} M)$ and m_i be the rank of $\mathrm{gr} M$ on Z_i. Then $SC(M) = \sum m_i Z_i$. We saw (by the Jantzen filtration argument) that $SC(M)$ does not depend on the choice of a good filtration. Moreover, if $0 \to M_1 \to M_2 \to M_3 \to 0$ is a short exact sequence of holonomic modules, then $SC(M_2) = SC(M_1) + SC(M_3)$.

<u>Corollary 3</u>. Holonomic $\mathcal{D}$-modules have finite length.

<u>Pf.</u> The length is $\leq \sum m_i$.

<u>Inverse and direct image functors.</u>

Let $\pi: X \to Y$ be a morphism of affine varieties. Then we have a pullback map for functions $\pi^*: \mathcal{O}(Y) \to \mathcal{O}(X)$.

We want to have a similar operation for $\mathcal{D}$-modules.

<u>Def.</u> Let M be a left $\mathcal{D}_Y$-module. The inverse image $\pi^{*0}(M)$ is the left $\mathcal{D}_X$-module $\pi^{*0}(M) = \mathcal{O}(X) \otimes_{\mathcal{O}(Y)} M$, with action of $\mathcal{D}_X$ defined by

$$v(f \otimes m) = vf \otimes m + f \otimes \pi_*(v) m.$$

It's easy to check that it is well defined and gives a structure of a $\mathcal{D}_X$-module.

<u>Rem.</u> Superscript 0 is used because this functor is "underived".

<u>Rem.</u> It is obvious that π^{*0} is right exact.

Let $\mathcal{D}_{X \to Y} = \pi^{*0} \mathcal{D}_Y = \mathcal{O}(X) \otimes_{\mathcal{O}(Y)} \mathcal{D}_Y$.

This is a left $\mathcal{D}_X$-module and a right $\mathcal{D}_Y$-module.

<u>Lemma 4.</u> $\pi^{*0} M = \mathcal{D}_{X \to Y} \otimes_{\mathcal{D}_Y} M$.

<u>Proof.</u> $\mathcal{D}_{X \to Y} \otimes_{\mathcal{D}_Y} M = \mathcal{O}(X) \otimes_{\mathcal{O}(Y)} \mathcal{D}_Y \otimes_{\mathcal{D}_Y} M = \mathcal{O}(X) \otimes_{\mathcal{O}(Y)} M$.

~~Corollary~~.

<u>Remark.</u> If $y_1, \dots, y_m$ is a coordinate system on Y and $\partial_1, \dots, \partial_m$ the corresponding vector fields

then $\mathcal{D}_{X\to Y} = \bigoplus_\alpha \mathcal{O}_X \partial^\alpha$.

<u>Direct image</u>. In analysis, for a map $\pi: X \to Y$ we have pushforward (integration by fibers) for distributions. We'd like to have a similar operation for $\mathcal{D}$-modules. This operation is more natural for right $\mathcal{D}$-modules, as measures (top forms) naturally form a right $\mathcal{D}$-module.

<u>Def</u>. For a morphism $\pi: X \to Y$ we define the functor $\pi_{*0}: \mathcal{M}^r(\mathcal{D}_X) \to \mathcal{M}^r(\mathcal{D}_Y)$ by $\pi_{*0}(M) = M \otimes_{\mathcal{D}_X} \mathcal{D}_{X\to Y}$.

<u>Ex.</u> $X = \{0\}$ and $Y = \mathbb{A}^1$, $\pi: X \to Y$. Consider k as a $\mathcal{D}_X$-module.

$$\pi_{*0} k = \mathcal{D}_{X\to Y} = \mathcal{O}_{(X)} \otimes_{\mathcal{O}_{(Y)}} \mathcal{D}_Y = \bigoplus_{n \geq 1} k\left(\frac{\partial}{\partial x}\right)^n = \mathcal{D}_{\mathbb{A}^1}/(x\mathcal{D}_{\mathbb{A}^1}) = \delta_0$$

as a right $\mathcal{D}$-module.

<u>Rem</u>. Since $\mathcal{M}^l \cong \mathcal{M}^r$, we can define the direct image functor on left $\mathcal{D}$-modules as well:

$$\pi_{*0}(M) = (M \otimes \Omega^n(X) \otimes_{\mathcal{D}_X} \mathcal{D}_{X\to Y}) \otimes \Omega^n(Y)^*.$$

Ex. $X = \mathbb{A}^1$, $Y = pt$. Then $\forall$ right $\mathcal{D}_X$-module M we have $\pi_{*0}(M) = M/(M\cdot\partial)$. Indeed, $\mathcal{D}_{X\to Y} = \mathcal{O}(X)\otimes_k k = \mathcal{O}(X)$, and $\mathcal{O}(X)$ is generated as a left $\mathcal{D}_X$-module by 1 with the defining relation $\partial\cdot 1 = 0$.

More generally, we have the following Lemma.

Lemma 5. Let $\pi : X \to pt$. Then for every right $\mathcal{D}_X$-module M, $\pi_{*0} M = M/M\cdot \mathrm{Vect}(X) = M \otimes_{\mathcal{D}_X} \mathcal{O}(X)$. (Coinvariants of the Lie algebra of vector fields). The proof is straightforward.

Remark. Analogously to the case of affine space, one can define the De Rham complex of a left $\mathcal{D}$-module N:

$$0 \to N \to N \otimes_{\mathcal{O}(X)} \Omega^1(X) \to N \otimes_{\mathcal{O}(X)} \Omega^2(X) \to \cdots \to N \otimes_{\mathcal{O}(X)} \Omega^n(X) \to$$

$$d(v\otimes\omega) = \nabla v \otimes \omega + v \otimes d\omega.$$

Its cohomology is denoted by $H^i_{dR}(N)$, and we see that for $\pi_0 : X \to pt$ $\pi_{*0} N = H^n_{dR}(N)$ (exercise).

Remark. Clearly π_0 is right exact.

For $\mathcal{D}$-modules there is an operation of tensor product, $M, N \to M \underset{\mathcal{O}(X)}{\otimes} N$ which we already considered for affine space. (As for affine space, a left $\mathcal{D}_X$-module = an $\mathcal{O}(X)$-module with a flat connection). Also there is an operation of external tensor product: if M is a $\mathcal{D}_X$-module and N a $\mathcal{D}_Y$-module then $M \boxtimes N = M \underset{k}{\otimes} N$ as a $\mathcal{D}_{X\times Y}$-module. Let $\Delta : X \to X \times X$ be the diagonal map.

<u>Lemma 6</u>. $M \underset{\mathcal{O}(X)}{\otimes} N = \Delta^{*0}(M \boxtimes N)$.

<u>Proof</u>. Easy.

<u>Remark</u>. This corresponds to the identity $f(x) g(y)\big|_{y=x} = (fg)(x)$ for functions.

<u>Theorem 7</u>. Let $X \xrightarrow{\pi} Y \xrightarrow{\tau} Z$ be morphisms of affine algebraic varieties. Then

(1) $(\tau \circ \pi)_{*0} = \tau_{*0} \circ \pi_{*0}$ and $(\tau \circ \pi)^{*0} = \pi^{*0} \circ \tau^{*0}$ (easy).

(2) π_{*0} and π^{*0} map holonomic modules to holonomic ones. The same is true for their derived functors (since π^{*0} and π_{*0} are right exact, $L^i \pi_{*0}$ and $L^i \pi^{*0}$ are defined)

We will prove this theorem later.

<u>Theorem 8</u> (Kashiwara). Let $i: X \to Y$ be a closed embedding. Then i_{*0} is an equivalence between $\mathcal{M}^r(\mathcal{D}_X)$ and $\mathcal{M}^r_X(\mathcal{D}_Y)$, where $\mathcal{M}^r_X(\mathcal{D}_Y)$ is the category of right $\mathcal{D}_Y$-modules M which are set-theoretically supported on X (i.e. for any $f \in \mathcal{O}(Y)$, $f|_X = 0$, and any $v \in M$, $\exists N$ s.t. $vf^N = 0$).

<u>Ex.</u> If $i : \{0\} \to \mathbb{A}^1$, then the theorem is saying that any $\mathcal{D}$-module M supported at 0 is $\delta_0 \otimes V$ for some vector space V. In this case, it's easy to see directly: Let $V \subset M$ be the kernel of x. Then we have an embedding $\delta_0 \otimes V \to M$. The $\mathcal{D}$-module $M/\delta_0 \otimes V$ is set-theoretically supported at 0, so it contains a copy of δ_0. But $\mathrm{Ext}^1(\delta_0, \delta_0) = 0$ (exercise), so for $\delta \in \delta_0 \subset M$, there is a lift $\tilde{\delta}$ such that $\tilde{\delta} x = 0$. This contradicts the def. of V.

Appendix to Lecture 8.

<u>Proposition 9.</u> If N is a left $\mathcal{D}_X$-module and $\pi : X \to pt$, then $\pi_{*i}(N) = H_{dR}^{n+i}(N)$.

<u>Proof.</u> $\pi_{*i}(N) = \mathrm{Tor}_{-i}^{\mathcal{D}_X}(N \otimes \Omega^n, \mathcal{O})$

$= \mathrm{Tor}_{-i}^{\mathcal{D}_X}(\Omega^n, N)$.

Recall that $dR(\mathcal{D}_X)$ is a resolution of Ω^n by free $\mathcal{D}_X$-modules:

$$\cancel{0 \to \mathcal{D}_X \to \mathcal{D}_X \otimes_{\mathcal{O}_{left}} \Omega^1 \to \cdots \to \mathcal{D}_X \otimes_{\mathcal{O}_{left}} \Omega^n \to 0}.$$

$$0 \to \mathcal{D}_X \to \Omega^1 \otimes_{\mathcal{O}} \mathcal{D}_X \to \cdots \to \Omega^n \otimes_{\mathcal{O}} \mathcal{D}_X \to 0.$$

So $\pi_{*i}(N)$ is the cohomology of the complex

$$0 \to N \to \Omega^1 \otimes_{\mathcal{O}} N \to \cdots \to \Omega^n \otimes_{\mathcal{O}} N \to 0,$$

which is $dR(N)$.