Lecture 8.

In more detail: (proof of lemma 9 from last lecture)
\( w \circ f v = -L_{f v} w = d \ i_{f v} w = -d (f \circ v \ w) = -d f v \ i_v w + f d i_v w = -L_v f \circ v \ w + f L_v w = f (L_v \ w) - L_v (f \circ v \ w) = f w \circ f v \)

Lemma 1. Let \( M \) be a left \( D_x \)-module. Then \( D_x \otimes M \) is a right \( D_x \)-module with
\( (m \otimes w) \circ v = -v m \otimes w - m \otimes L_v (w) \)

Proof. First show it's well defined. For this we need to show that
\( (f m \otimes w - m \otimes f w) \circ v = 0 \).

We have
\( (f m \otimes w - m \otimes f w) \circ v = -v (f m) \otimes w + v m \otimes f w \)
\(- f m \otimes L_v w + f w \otimes L_v (f w) \)
\(- L_v f \otimes m \otimes w - f w m \otimes w + v m \otimes f w \)
\(- f m \otimes L_v w + m \otimes f w \).

Also clearly
\( (m \otimes w) [v, w] = (m \otimes w) v \otimes w \)
\(- (m \otimes w) w \otimes v \), and
\( (m \otimes w) \circ f v = f (m \otimes w) \circ v \) (as follows from the previous lemma).
Remark. Similarly, if $\mathcal{M}$ is a right $D_x$-module then $\mathcal{M} \otimes \mathcal{O}^n(x)^* \otimes \mathcal{O}^n(x)$ is a left $D_x$-module.

Corollary 2. The categories $\mathcal{M}(D_x)$ and $\mathcal{M}(D_x^\mathfrak{c})$ are canonically equivalent. The equivalence is given by $\mathcal{M} \mapsto \mathcal{M} \otimes \mathcal{O}^n(x)$.

So we will not make a distinction between these categories and call the category of $D_x$-modules $\mathcal{M}(D_x)$.

Remark. The algebras $D_x$ and $D_x^\mathfrak{c}$ are not necessarily isomorphic, but they are canonically Morita equivalent.

Let $\mathcal{M}$ be a finitely generated $D_x$-module. Then it has a good filtration $F$ for the geometric filtration on $D_x$ (which is the only one available for general $X$) and $\text{supp}(\text{gr}_F\mathcal{M}) = \text{SS}(\mathcal{M})$.

Filtration. $\text{SS}(\mathcal{M})$ is a subvariety of $T^*X$ called the regular support of $\mathfrak{m}$. By Gabber's theorem, its support is independent of $\mathfrak{m}$.

By the coinstantic condition, $\dim \text{SS}(\mathcal{M}) = n$. 
If \( \dim SS(M) = n \) \( (\Rightarrow SS(M) \) is Lagrangian), then we say that \( M \) is holonomic.

Assume \( M \) is holonomic. Then we can define a cycle (the singular cycle of \( M \)) in the following way: let \( Z_1, \ldots, Z_k \) be the irreducible components of \( \text{supp}(\text{gr} M) \) and \( m_i \) be the rank of \( \text{gr} M \) on \( Z_i \). Then \( SC(M) = \sum m_i Z_i \). We say (by the Jantzen filtration argument) that \( SC(M) \) does not depend on the choice of a good filtration. Moreover, if \( 0 \to M_1 \to M_2 \to M_3 \to 0 \) is a short exact sequence of holonomic modules, then \( SC(M_2) = SC(M_1) + SC(M_3) \).

**Corollary 3.** Holonomic D-modules have finite length.

**Pt.** The length is \( \leq \sum m_i \).

*Inverse and direct image functors.* Let \( \pi: X \to Y \) be a morphism of affine varieties. Then we have a pullback map for functions \( \pi^* \mathcal{O}(Y) \to \mathcal{O}(X) \).
We want to have a similar operation for $D$-modules.

**Def.** Let $M$ be a left $D_y$-module. The inverse image $\pi^*(M)$ is the left $D_x$-module $\pi^*(M) = \mathcal{O}(x) \otimes \mathcal{O}(y) M$, with action of $D_x$ defined by

$$v(f \otimes m) = vf \otimes m + f \otimes \pi^*(v)m.$$  

It's easy to check that it is well defined and gives a structure of a $D_x$-module.

**Rem.** Superscript 0 is used because this functor is "undervived".

**Rem.** It is obvious that $\pi^0$ is right exact.

Let $D_x \to y = \pi^0 D_y = \mathcal{O}(x) \otimes \mathcal{O}(y) D_y$. This is a left $D_x$-module and a right $D_y$-module.

**Lemma 4.** $\pi^0 M = D_x \to y \otimes D_y M$.

**Proof.** $D_x \to y \otimes D_y M = \mathcal{O}(x) \otimes D_x \otimes D_y M = \mathcal{O}(x) \otimes M$.

**Remark.** If $y_1, \ldots, y_m$ is a coordinate system on $x$, and $\partial_1, \ldots, \partial_m$ are the corresponding vector fields.
then $D_x \to y = \bigoplus_{x} D_x x$

**Direct image.** In analysis for a map $\pi : X \to Y$ we have pushforward for distributions. We'd like to have a similar operation for $D$-modules. This operation is more natural for right $D$-modules, as measures (top forms) naturally form a right $D$-module.

**Def.** For a morphism $\pi : X \to Y$ we define the functor $\pi_* : \mathcal{M}^r(D_x) \to \mathcal{M}^r(D_y)$ by $\pi_* (M) = M \otimes D_x \to y$.

**Ex.** $X = \{ 0, y \}$ and $Y = \mathbb{A}^1$, $\pi : X \to Y$. Consider $k$ as a $D_x$-module.

$\pi_* k = D_x \to y = k(0) \otimes D_y = \bigoplus_{n=1} k(\frac{\partial}{\partial x})^n = \mathcal{D}_1/(x \mathcal{D}_1) = \delta_0$

as a right $D$-module.

**Rem.** Since $\mathcal{M}^r = \mathcal{M}^r$, we can define the direct image functor on left $D$-modules as well:

$\pi_* \mathcal{M}^r = (M \otimes \mathcal{O}(x) \otimes D_x \to y) \otimes \mathcal{O}^r(y)$.
Ex: \( x = A^1 \), \( y = \text{pt} \). Then a right \( D_x \)-module \( M \) we have \( \mathcal{T}_0(M) = M / (M, \partial) \). Indeed, \( D_x \to y = \mathcal{O}(x) \otimes k = \mathcal{O}(x) \), and \( \partial(x) \) is generated as a left \( D_x \)-module by \( 1 \) with the defining relation \( \partial \cdot 1 = 0 \).

More generally, we have the following lemma.

**Lemma 5.** Let \( \pi : X \to \text{pt} \). Then for every right \( D_x \)-module \( M \), \( \mathcal{T}_0 M = M / (M \cdot \text{Vect}(x)) = M \otimes \mathcal{O}(x) \) \((\text{coinvariants of the Lie algebra of vector fields})\). The proof is straightforward.

**Remark.** Analogously to the case of affine space, one can define the De Rham complex of \( M \), a left \( D \)-module \( N \):

\[
0 \to N \to N \otimes \Omega^1(x) \to N \otimes \Omega^2(x) \to \cdots \to N \otimes \Omega^n(x) \to 0
\]

\( d(v \otimes \omega) = \partial(v \otimes \omega) + v \otimes d \omega \).

Its cohomology is denoted by \( H^i_{\text{dR}}(N) \), and we see that for \( \pi_0 : X \to \text{pt} \)

\( \mathcal{T}_0 N = H^n_{\text{dR}}(N) \) (exercise).

**Remark.** Clearly \( \mathcal{T}_0 \) is right exact.
For $D$-modules there is an operation of tensor product $\otimes$, $M, N \to M \otimes N$ which we already considered for affine space. (As for affine space, a left $D_x$-module = an $O(x)$-module with a flat connection). Also there is an operation of external tensor product: if $M$ is a $D_x$-module and $N$ a $D_y$-module then $M \otimes N = M \otimes N$ as a $D_{x \times y}$-module.

Let $\Delta : X \to X \times X$ be the diagonal map.

Lemma 6. $M \otimes N = \Delta^* (M \otimes N)_{O(x)}$.

Proof. Easy.

Remark. This corresponds to the identity $f(x)g(y)|_{y = x} = (fg)(x)$ for functions.

Theorem 7. Let $X \xrightarrow{\pi} Y \xrightarrow{\pi} Z$ be morphisms of affine algebraic varieties. Then

1. $(\pi_\circ \pi_\circ)^* = \pi_\circ \circ \pi_\circ$ and $(\pi_\circ \circ \pi_\circ)^* = \pi_\circ \circ \pi_\circ$ (easy).
2. $\pi_\circ$ and $\pi_\circ^\circ$ map holonomic modules to holonomic ones. The same is true for their derived functors (since $\pi_\circ$ and $\pi_\circ^\circ$ are right exact, $L^i \pi_\circ$ and $L^i \pi_\circ^\circ$ are defined).
We will prove this theorem later.

Theorem 8. Let $i^*: X \to Y$ be a closed embedding. Then $i^*$ is an equivalence between $\mathcal{M}^r(D_x)$ and $\mathcal{M}^r(D_y)$, where $\mathcal{M}^r(D_x)$ is the category of $\mathcal{D}_x$-modules $M$ which are set-theoretically supported on $X$ (i.e. for any $f \in \mathcal{O}(Y)$, $f|_X = 0$, and any $v \in M$, $\exists N$ s.t. $\text{supp}^n_v = 0$).

\textbf{Ex.} If $i: \mathcal{O}_Y \to \mathcal{A}_Y$, then the theorem is saying that any $\mathcal{D}_x$-module $M$ supported at $0$ is $\mathcal{O}_Y \otimes V$ for some vector space $V$. In this case, it's easy to see directly:

Let $V \subset M$ be the kernel of $x$. Then we have an embedding $\mathcal{O}_Y \otimes V \to M$. The $\mathcal{D}_x$-module $M/\mathcal{O}_Y \otimes V$ is set-theoretically supported at $0$, so it contracts a copy of $\mathcal{O}_Y$. But $\text{Ext}^1(\mathcal{O}_Y, \mathcal{O}_Y) = 0$ (exercise), so for $\mathcal{O}_Y \subset M$, there is a lift $\delta$ such that $\delta x = 0$. This contradicts the left of $\mathcal{O}_Y$. 

Proposition 9. If \( N \) is a left \( D_x \)-module and \( \pi : X \to \text{pt} \), then \( \Pi_x : (N) = H_{dR}^{n+i}(N) \).

Proof. \( \Pi_x (N) = \text{Tor}^{D_x}_{-i} (N \otimes S^n, 0) \)
\[
= \text{Tor}^{D_x}_{-i} (S^n, N).
\]

Recall that \( dR(D_x) \) is a resolution of \( S^n \) by free \( D_x \)-modules:
\[
0 \to D_x \to S^1 \otimes D_x \to \cdots \to S^n \otimes D_x \to 0.
\]
So \( \Pi_x (N) \) is the cohomology of the complex
\[
0 \to N \to S^1 \otimes N \to \cdots \to S^n \otimes N \to 0,
\]
which is \( dR(N) \).