Lecture 7.

D-modules on general affine varieties

Let X be an affine algebraic variety over a field k, char $k=0$. We'd like to define the algebra of differential operators on X, denoted $\mathcal{D}_X$.

Def. (Grothendieck). Let R be a commutative ring or k-algebra. Let $\mathcal{D}_{-1}(R)=0$, and for $n\geq 0$
$$\mathcal{D}_n(R)=\{L:R\to R \mid [L,f]\in\mathcal{D}_{n-1}(R)\ \forall f\in R\}$$
(where f stands for the operator of multiplication by f). Elements of $\mathcal{D}_n(R)$ are called differential operators on R of order n.

Let $\mathcal{D}(R)=\bigcup_n \mathcal{D}_n(R)$. (clearly $\mathcal{D}_n(R)\subset\mathcal{D}_{n+1}(R)$)

Note that $\mathcal{D}_0(R)=R$, $\mathcal{D}_n(R)\cdot\mathcal{D}_m(R)\subset\mathcal{D}_{n+m}(R)$, $[\mathcal{D}_n(R),\mathcal{D}_m(R)]\subseteq\mathcal{D}_{n+m-1}(R)$.

(this is proved by induction in $m+n$).

So $\{\mathcal{D}_n\}$ is an algebra filtration of $\mathcal{D}$, which is an algebra containing R, and $\mathrm{gr}\mathcal{D}$ is commutative.

Ex. If $R=k[x_1,\dots,x_n]$ then $\mathcal{D}(R)=\mathcal{D}(\mathbb{A}^n)$ (we will see it later).

Ex. $\mathcal{D}_1(R)=\mathrm{Der}(R)\oplus R$. Indeed, if $L\in\mathcal{D}_1(R)$ then $\forall f\in R$ $[L,f]\in R$, and
$$[L,fg]=[L,f]g+f\cdot[L,g],$$
so $\mathrm{ad}L: R\to R$.

~~all~~ $\bar{L} \in \operatorname{Der} R$. Thus

$$Lf = \underset{\operatorname{Der} R}{\bar{L} f} + \underset{R}{L(1) f}, \text{ so } \mathcal{D}_1(R) = \operatorname{Der} R \oplus R.$$

But $\mathcal{D}_2(R)$ is in general not $\mathcal{D}_1(R) \oplus$ smth natural.

<u>Def.</u> Let M, N be R-modules. We define $\mathcal{D}_{-1}(M,N) = 0$, and for $n \geq 0$,

$$\mathcal{D}_n(M,N) = \{ \underset{\text{linear map}/k}{L : M \to N} \mid [L, f] \in \mathcal{D}_{n-1}(M,N) \ \forall f \in R \}. \quad \mathcal{D}(M,N) = \bigcup_n \mathcal{D}_n(M,N).$$

We say that $L \in \mathcal{D}_n(M,N)$ is a differential operator from M to N of order $\leq n$.

We set $\mathcal{D}_X = \mathcal{D}(\mathcal{O}_X)$.

This definition makes sense for any X but if X is singular, $\mathcal{D}_X$ has bad properties (i.e. it may not be Noetherian) so we'll work with $\mathcal{D}_X$ only if X is smooth, of dim $= n$. (X irreducible)

Let $U \subset X$ be an affine open set

<u>Def.</u> ~~A coordinate system on U is~~

$\mathcal{D}_{X,U,n} = \mathcal{D}_n(\mathcal{O}(X), \mathcal{O}(U))$, where $\mathcal{O}(X), \mathcal{O}(U)$ are $\mathcal{O}(X)$-modules.

Lemma 1. Let $L \in \mathcal{D}_{X,U,n}$ and $f \in \mathcal{O}(X)$ be such that $f(z) = df(z) = \dots = d^n f(z) = 0$ for some $z \in \cancel{X}\, U$. Then $(Lf)(z) = 0$. Conversely, if this property holds for $L : \mathcal{O}(X) \to \mathcal{O}(U)$ then $L \in \mathcal{D}_{X,U,n}$.

Proof: $\Rightarrow$ Induction in n. Base $(n=0)$ is clear, as $\mathcal{D}_{X,U,0} = \mathcal{O}(\cancel{X}\, U)$. Suppose the statement is known for $n-1$, prove it for n.

~~Let $x_1, \dots, x_n$ be local coordinates~~

We have $f \in m_z^{n+1}$, where $m_z \subset \mathcal{O}_X$ is the maximal ideal of z.

Thus we may assume that

$f = f_1 \cdots f_{n+1}$, $f_i \in m_z$.

We have

$$L f_1 \cdots f_{n+1} = f_{n+1} L f_1 \cdots f_n + [L, f_{n+1}] f_1 \cdots f_n.$$

The first term is zero at z, as $f_{n+1}(z) = 0$, and the second term is zero by the induction assumption.

Conversely, if property holds for L, let $f \in \mathcal{O}(X)$. Use induction in n.

Let $z \in X$, $f_1, \dots, f_n \in m_z$. Then

$$([L,f] f_1 \cdots f_n)(z) = ([L, f - f(z)] f_1 \cdots f_n)(z) =$$

$$(L (f - f(z)) f_1 \cdots f_n)(z) - (f - f(z)) L f_1 \cdots f_n (z) = 0.$$

So $[L, f] \in \mathcal{D}_{X,U,n-1}$ by the induction assumption.

Lemma 1 implies that given $L \in \mathcal{D}_{X,U,n}$, $\forall z \in U$ we can define a linear function $f \mapsto (Lf)(z)$ on $m_z^n / m_z^{n+1} = S^n T_z^* X = S^n T_z^* U$. This depends regularly on z, and thus defines a section $\sigma_n(L) \in \Gamma(U, S^n TU)$. Moreover, by Lemma 1, $\sigma_n |_{\mathcal{D}_{X,U,n-1}} = 0$, so σ_n gives rise to a linear map

$$\sigma_n : gr_n \mathcal{D}_{X,U} \to \Gamma(U, S^n TU), \quad \sigma : gr \mathcal{D}_{X,U} \to \Gamma(U, STU) \cong \mathcal{O}(T^*U)$$

$$\sigma = \oplus_n \sigma_n .$$

<u>Definition</u>. σ_n is called the symbol map.

Clearly, if $X = U$, $\sigma = \oplus_{n \geq 0} \sigma_n$ is an algebra map. in general it is a module map (when we regard $\mathcal{D}_{X,U}$ as a $\mathcal{D}_{U,U}$-module). Also,

<u>Lemma 2</u>. The map σ is an isomorphism.

<u>Proof</u>. σ is injective by Lemma 1: if $\sigma_n(L) = 0$ then $L \in \mathcal{D}_{X,U,n-1}$. Let us show that σ is surjective. It suffices to do so for $U = X$, as we have a natural map $\varphi : \mathcal{O}(U) \otimes_{\mathcal{O}(X)} \mathcal{D}_{X,n} \to \mathcal{D}_{X,U,n}$, and $\Gamma(U, S^n TU) = \mathcal{O}(U) \otimes_{\mathcal{O}(X)} \Gamma(X, S^n TX)$. But for $U = X$ σ_0 and σ_1 are isomorphisms, and the algebra $\Gamma(X, STX) = \mathcal{O}(T^*X)$ is generated in degree 1 (it's true for $\Gamma(X, SE)$, where E is any vector bundle). So the statement follows from the fact that σ is an algebra map for $X = U$ and a module map in general.

<u>Corollary 3</u>. $\mathcal{D}_X$ is generated as an algebra by $\mathcal{O}(X)$ and $\mathrm{Vect}(X)$, i.e. by $\mathcal{D}_{X,1}$.

<u>Pf.</u> This is true by lemma 2 for $\mathrm{gr}\,\mathcal{D}_X$, so true for $\mathcal{D}_X$.

<u>Corollary 4</u>. The restriction map

$$r: \mathcal{D}_{V,U} \longrightarrow \mathcal{D}_{X,U} \quad \text{for} \quad X \supset V \supset U \; \left(V \text{ open affine}\right)$$

is an isomorphism. Thus, $\mathcal{D}_{X,U} \underset{\text{naturally}}{\cong} \mathcal{D}_U = \mathcal{D}_{U,U}$.

<u>Proof</u>. The map $\mathrm{gr}(r)$ is an isomorphism by lemma 2.

<u>Corollary 5</u>. The natural map

$$\varphi: \mathcal{O}(U) \otimes_{\mathcal{O}(X)} \mathcal{D}_X \longrightarrow \mathcal{D}_{X,U} \cong \mathcal{D}_U$$

is an isomorphism.

<u>Proof.</u> By lemma 2 $\mathrm{gr}(\varphi)$ is an isomorphism. This shows that $\mathcal{D}_X$ is a quasicohent sheaf on X, with $\Gamma(U, \mathcal{D}_X) = \mathcal{D}_U$.

Def. A coord. system on an open set $U \subset X$ is a collection of vector fields $\partial_1, \dots, \partial_n$ and a collection of functions $x_1, \dots, x_n$ on U such that $\partial_i(x_j) = \delta_{ij}$. Such a system is clearly the same thing as an etale map $U \to \mathbb{A}^n$ (i.e. a map whose differential is everywhere an isomorphism.

<u>Prop 2.</u> ~~On~~ For each smooth X ~~on~~ and $x \in X$, $\exists U \ni x$ such that there is a coordinate system on U.

<u>Proof.</u> Let $x_1, \dots, x_n : X \to k$ be any regular functions, such that $dx_1, \dots, dx_n$ are ~~generically~~ linearly independent ~~take~~ at x. Then take $U = \{y \mid dx_1, \dots, dx_n$ are lin. independent ~~at~~ $y\}$ ∎

So assume that there is a coordinate system on the whole X.

<u>Lemma 6</u> $\mathcal{D}_{X,n} \cong \bigoplus_{\alpha : |\alpha| \le n} \mathcal{O}_X \partial^\alpha$, where for $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}_+^n$, $|\alpha| = \alpha_1 + \dots + \alpha_n$, $\partial^\alpha \overset{\text{def}}{=} \partial_1^{\alpha_1} \dots \partial_n^{\alpha_n}$

~~Proof. By induction in order of diff. oper.~~

Pf.
We have a natural map of algebras

$$\psi: \bigoplus_{\alpha} \mathcal{O}_X \partial^\alpha \longrightarrow \mathcal{D}_X.$$

By lemma 2, this map defines an isomorphism at the graded level. So it's an isomorphism

Corollary 7. The isomorphism $\sigma: \mathrm{gr}\,\mathcal{D}_X \to \mathcal{O}(T^*X)$ for any affine X is an isomorphism of Poisson algebras, if the Poisson bracket on T^*X is defined by its symplectic form.

Proof. It suffices to check it when X has a coordinate system, in which case it's easy.

Corollary 8. $\mathcal{D}_X$ is generated by $\mathcal{O}(X)$ and D_v, $v \in \mathrm{Vect}\,X$, subject to the following relations:

1) $[D_v, f] = v(f)$, $v \in \mathrm{Vect}\,X$, $f \in \mathcal{O}(X)$.

2) $[D_v, D_w] = D_{[v,w]}$. 3) $D_{fv} = fD_v$.

Proof. These relations are clearly satisfies. Let $\tilde{\mathcal{D}}_X$ be the algebra defined by these relations. We have a surjective homomorphism $\varphi: \tilde{\mathcal{D}}_X \to \mathcal{D}_X$, ~~The map $\tilde{\mathcal{D}}_{X,n} \to \mathcal{D}_{X,n}$ is a morphism of coherent sheaves, so~~ which defines the map of graded algebras $\mathrm{gr}\,\tilde{\mathcal{D}}_X \to \mathcal{O}(T^*X)$.

Remark. The above implies that $\mathcal{D}_{X,n}$ is a projective $\mathcal{O}(X)$-module of finite rank, and $\mathcal{D}^\vee_{X,n} = \mathrm{Hom}_{\mathcal{O}(X)}(\mathcal{D}_{X,n}, \mathcal{O}(X)) \cong \mathcal{O}(X^{(n)})$, where $X^{(n)}$ is the n-th nilpotent neighborhood of the diagonal in $X \times X$, i.e. $\mathcal{O}(X^{(n)}) = \mathcal{O}(X) \otimes \mathcal{O}(X) / m_\Delta^{n+1}$, m_Δ - ideal of the diagonal.

So $\mathcal{D}^\vee_X = \varprojlim_{n \to \infty} \mathcal{O}(X^{(n)}) = \mathcal{O}(X^{(\infty)})$, where $X^{(\infty)}$ is the formal neighborhood of the diagonal in $X \times X$.

Note that $\mathrm{gr}\, \mathcal{O}(X^{(\infty)}) = \mathcal{O}(TX)$, which agrees with the fact that $\mathrm{gr}\, \mathcal{D}_X = \mathcal{O}(T^*X)$.

The algebra $gr\,\tilde{D}_X$ is defined by generators L_v over $\mathcal{O}_X$, with relations

1) $[D_v, f]=0$
2) $[D_v, L_w]=0$
3) $D_{fv}=fD_v$,

and posibly some others. But already these relations define $\mathcal{O}(T^*X)$. So there is no other relations, and ψ is an isomorphism. (8a)

<u>Left and right $\mathcal{D}$-modules</u>

Let $\mathcal{M}^\ell(\mathcal{D}_X)$ be the category of left $\mathcal{D}_X$-modules and $\mathcal{M}^r(\mathcal{D}_X)$ the category of right $\mathcal{D}_X$-modules.

By Corollary 8, a left $\mathcal{D}_X$-module structure on M (an $\mathcal{O}_X$-module) is a Lie algebra action of $Vect(X)$ on M such that $v(fm) - f\,vm = v(f)m$, $(fv)(m) = f\cdot v(m)$.

Similarly, a right $\mathcal{D}_X$-module structure is the same, except the action of $Vect(X)$ should be an antihomomorphism and $m(vf)=f\cdot(mv)=mv\cdot f$.

<u>Lemma 9</u>. Ω_X (top differential forms) has a canonical structure of a right $\mathcal{D}$-module.

<u>Proof</u>. Let $v(\omega)=-L_v\omega$. Then it satisfies the required conditions. (exercise)

Appendix to lecture 7.

Differential operators in characteristic p.

Let $R = k[x]$, where k has characteristic p. Let us compute $\mathcal{D}(R)$.

<u>Claim</u>: $\mathcal{D}_n(R)/\mathcal{D}_{n-1}(R) \cong k[x]\frac{\partial^n}{n!}$ ~~is spanned by~~

<u>Proof</u>. For each z, let $L \in \mathcal{D}_n(R)$, consider

$L(x-z)^n|_{x=z}$. This is a regular function, call it f ($f \in k[x]$). Then $L - f(x)\frac{\partial^n}{n!} \in \mathcal{D}_{n-1}$.

<u>Corollary</u>: $\mathcal{D}(R) = \{\sum_i f_i(x)\frac{\partial^i}{i!}\}$ (differential operators with divided powers). More generally,

$$\mathcal{D}(k[x_1,\dots,x_n]) = \Big\{\sum_{\alpha=(\alpha_1,\dots,\alpha_n)} f_\alpha(x)\frac{\partial^\alpha}{\alpha!}\Big\},$$

$\frac{\partial^\alpha}{\alpha!} = \frac{\partial_1^{\alpha_1}}{\alpha_1!}\cdots\frac{\partial_n^{\alpha_n}}{\alpha_n!}$. This will be denoted $\mathcal{D}_{dp}(\mathbb{A}^n)$.

This shows that in characteristic p, $\mathcal{D}_{dp}(\mathbb{A}^n)$ is <u>not</u> generated by $\mathcal{O}$ and Vect, i.e. Cor 3 of lecture 7 is false in char p. It's also false that σ is an isomorphism (in fact, σ isn't even defined!). Indeed, $L \in \mathcal{D}_n$ gives rise to an element $\bar{L}$ of $(m_z^n/m_z^{n+1})^* = (S^n T_z X)^*$, so to define $\sigma_n(L)$, we need to compose with an isomorphism

$(S^n T_z^* X)^* \xrightarrow{\gamma_n} S^n T_z^* X$, i.e. $\sigma_n(L) = \gamma_n(\bar{L})$.
The map γ_n^* corresponds to a pairing between $S^n V$ and $S^n V^*$, $V = T_z X$, given by
$(f, g) = f(\partial) g(x)|_{x=0}$. This pairing is nondegenerate only in characteristic zero. And γ_n is obtained by inversion of the pairing, so it is not well defined. In fact, $\mathrm{gr}\, \mathcal{D}_{dP}(A^n)$ is an infinitely generated algebra. E.g. for $n=1$,
$\mathcal{D}_{dP}(A^1) = k[x, \frac{\xi^j}{j!}, j \geq 1]$. It has basis
$\frac{\xi^j}{j!} = a_j$ over $k[x]$ with multiplication law $a_j \cdot a_r = \binom{j+r}{j} a_{j+r}$.

One can also define $\mathcal{D}_{dP}(X) = \mathcal{D}(\mathcal{O}(X))$ for any smooth affine variety over k. One can show as in characteristic zero that modules over $\mathcal{D}_{dP}(X) = \mathcal{D}(\mathcal{O}(X))$ are the same as !-crystals on X.
Note that the subalgebra of $\mathcal{D}(\mathcal{O}(X))$ generated by $\mathcal{O}(X)$ and $\mathrm{Vect}\, X$ is actually quite small.

Namely, e.g. for $X=\mathbb{A}^1$, $\langle \mathcal{O}, Vect\rangle \subset \mathcal{D}_{dP}$ is the set of operators $f_0(x)+f_1(x)\partial+\dots+f_{p-1}(x)\partial^{p-1}$, as $\partial^p=0$. So it is finite over $\mathcal{O}$ (of rank p).

One can, however, define another version of differential operators on X, called crystalline differential operators, for which the symbol map is defined and is an isomorphism. This is done using the generators and relations as in char. 0: $\mathcal{D}_{cr}(X)$ is generated by $\mathcal{O}(X)$ and $Vect(X)$ with relations:

1) $[D_v, f] = L_v(f)$
2) $[D_v, D_w] = D_{[v,w]}$
3) $D_{fv} = fD_v$.

Then $\mathcal{D}_{cr}(X)$ has a filtration and $gr\,\mathcal{D}_{cr}(X) \cong \mathcal{O}(T^*X) = \bigoplus_n \Gamma(X, S^nTX)$.

This can be shown e.g. by using local coordinates. However, the action of $\mathcal{D}_{cr}(X)$ on $\mathcal{O}(X)$ is not faithful, i.e. the natural map $\mathcal{D}_{cr}(X) \to \mathcal{D}_{dP}(X)$ is neither surjective nor injective.