Lecture 6.

Functional dimension and homological algebra.

Let M be a f.g. $\mathcal{D}$-module

$d_a(M) = d(M) = \dim SS_a(M)$ and $d_g(M) = \dim SS(M)$.

Theorem 1 $d_a(M) = d_g(M)$.

To prove this theorem, let's develop some homological technology.

Theorem 2 Let A be a filtered algebra over k such that $\mathrm{gr} A$ is a finitely generated commutative regular algebra of $\dim = m$ (i.e., $\mathrm{gr} A = \mathcal{O}(X)$, where X is a smooth irred. alg. variety) of $\dim = m$.
Let M be a f.g. A-module. Let $d(M) = \dim(\mathrm{supp}\, \mathrm{gr}^F M)$, where F is a good filtration on M, and $j(M) = \min\{j \mid Ext^j(M,A) \neq 0\}$.
Then

(1) $d(M) + j(M) = m$

(2) $Ext^j(M,A)$ is a finitely generated right A-module, and $d(Ext^j(M,A)) \leq m-j$

(3) For $j = j(M)$, we have an equality in (2).

Before giving a proof, we derive some corollaries.

Cor 2a. $d_a(M) = d_g(M) = 2n - j(M)$ for any f.g. M. $\Rightarrow$ Th 1.

From now on we'll write $d(M)$ for $d_a(M) = d_g(M)$.

<u>Cor 3</u>. For any f.g. $\mathcal{D}$-modules M, N, $\mathrm{Ext}^j(M,N)=0$ if $j>n$. Thus, the homological dimension of $\mathcal{D}$ is n.

<u>Proof</u>. Let P be a free f.g. module and $0\to K\to P\to N\to 0$ be a short exact sequence. Then we have

$$\mathrm{Ext}^j(M,P)\to \mathrm{Ext}^j(M,N)\to \mathrm{Ext}^{j+1}(M,K)\to \mathrm{Ext}^{j+1}(M,P)$$

But $\mathrm{Ext}^j(M,P)=\mathrm{Ext}^{j+1}(M,P)=0$ by Thm 2 (2) and the Bernstein inequality.
~~Is other sta~~
Also K is finitely generated, as $\mathcal{D}$ is Noetherian, and $\mathrm{Ext}^i=0$ for $i>2n$ since $\mathrm{gr}\,\mathcal{D}$ has homological dim $=2n$. Thus we get the desired statement by induction in $\ell=2n-j+1$ (starting from $\ell=0$).

<u>Cor 4.</u> M is holonomic $\Leftrightarrow$ $Ext^j(M, \mathcal{D}) \neq 0$ only if $j = n$.

<u>Proof.</u> M is holonomic $\Rightarrow$ $d(M) = n$ for Bernstein filtration $\Rightarrow$ $j(M) = n$ by (1) in Thm 2. So we have $Ext^j(M, \mathcal{D}) = 0$ for $j < n$ by def of $j(M)$. For $j > n$ $Ext^j(M, \mathcal{D}) = 0$ by Thm 2(2), as $m = 2n$, so $d(Ext^j(M, \mathcal{D})) \leq 2n - j$, and thus the statement holds by the Bernstein inequality. ∎

So the only Ext which has a chance of being nonzero is $Ext^n(M, \mathcal{D})$. This is a right $\mathcal{D}$-module, but we can turn it into a left $\mathcal{D}$-module by using the antiautomorphism $\sigma(x_i) = x_i$, $\sigma(\partial_i) = -\partial_i$.

<u>Cor 5</u> Let $\mathbb{D}(M) = Ext^n(M, \mathcal{D})$, considered as a left module. Then the functor $M \mapsto \mathbb{D}(M)$ is an exact contravariant functor on the category of holonomic $\mathcal{D}$-modules, and $\mathbb{D}^2 \cong id$.

Proof. Exactness of $\mathbb{D}$ follows immediately from the long exact sequence of Exts. Let us show that $\mathbb{D}^2 = id$. Let P be a f.g. projective $\mathcal{D}$-module. Let $P^\vee = Hom(P, \mathcal{D})$ be the dual module (considered as a left $\mathcal{D}$-module as before). Clearly $P^\vee$ is projective. If P° is a complex of projective $\mathcal{D}$-modules then denote by $(P^\vee)^\circ$ the complex defined by $(P^\vee)^i = (P^{-i})^\vee$.

Let M be a holonomic $\mathcal{D}$-module and let P° be its projective resolution (of length n, exists by 623). Then it's clear that $(P^\vee)^\circ[n]$ (shift by n) is a projective resolution of $\mathbb{D}(M)$. This implies that $\mathbb{D}^2 = id$.

Cor. 6. Let M be $\mathcal{O}$-coherent. Then $\mathbb{D}(M) = Hom_{\mathcal{O}}(M, \mathcal{O}) = M^\vee$ as an $\mathcal{O}$-module (i.e. $\mathbb{D}(M)$ is the dual vector bundle). The dual connection is described as follows: it's the only connection making the pairing between M and $M^\vee$;

invariant

if $m \in M$ and $m^\vee \in M^\vee$ then

$$(\nabla m, m^\vee) + (m, \nabla m^\vee) = d(m, m^\vee).$$

So if $\nabla_M = d + \omega$ then $\nabla_{M^\vee} = d - \omega^\vee$.

Before proving this corollary, let us develop some technology.

__Def.__ Let M be any $\mathcal{D}$-module. The De Rham complex of M is the complex

$$0 \to \underset{-n}{M} \to \underset{-n+1}{\Omega^1 \otimes_{\mathcal{O}} M} \to \cdots \to \cdots \underset{0}{\Omega^n \otimes_{\mathcal{O}} M} \to 0$$

(the second row shows the cohomological degrees). The differentials $d: \Omega^i \otimes_{\mathcal{O}} M \to \Omega^{i+1} \otimes_{\mathcal{O}} M$ are defined by the formula

$$d(\omega \otimes m) = d\omega \otimes m + (-1)^{|\omega|} \omega \wedge \nabla m.$$

(Leibniz rule). It's easy to check that it is well defined, and $d^2 = 0$. Moreover, if $M = \mathcal{O}$, this coincides with the usual De Rham complex.

(if M is a vector bundle, this is known as the De Rham complex with twisted coefficients). If $\nabla = d + \omega$, and $d\omega + [\omega, \omega] = 0$, then the differential $d_\omega = d + \omega\wedge$ satisfies $d_\omega^2 = 0$. This is exactly the differential in the above complex.

Notation. This complex denoted by $dR(M)$. In particular, we can consider the complex $dR(\mathcal{D})$, which is a complex of free right $\mathcal{D}$-modules.

Lemma 7. $H^i(dR(\mathcal{D})) = \begin{cases} 0, & i \neq 0 \\ \Omega^n, & i = 0. \end{cases}$

(here Ω^n is a right $\mathcal{D}$-module via $\alpha dv \cdot L = (L^* \alpha) dv$, where L^* is the adjoint of L under the integration pairing: $\partial_i^* = -\partial_i$, $x_i^* = x_i$. (i.e. $L^* = \sigma(L)$).

Proof. The complex $dR(\mathcal{D})$ looks like:

$$0 \to \mathcal{D} \to k^n \underset{k}{\otimes} \mathcal{D} \to \Lambda^2 k^n \otimes \mathcal{D} \to \cdots \to \Lambda^n k^n \otimes \mathcal{D} \to 0$$

with differential given by

$$d(\alpha \otimes L) = \sum_i \alpha \wedge dx_i \otimes \partial_i L. \qquad (\alpha \in \Lambda^i k^n, L \in \mathcal{D})$$

So this is the Koszul complex of $\mathcal{D}$ as a $k[\partial_1, \dots, \partial_n]$-module (but written backwards)

Thus, since $\mathcal{D}$ is a free module over $k[\partial_1, .., \partial_n]$, we get that the cohomology is nontrivial only ~~in~~ in degree 0, and is $\Lambda^n k^n \otimes_{(\partial_1,\dots,\partial_n)\mathcal{D}} \backslash \mathcal{D} = \Omega^n$ as a right $\mathcal{D}$-module.

(Recall that the Koszul complex of a module N over $k[t_1, .., t_n]$ is

$$0 \to \Lambda^n k^n \otimes N \to \dots\dots \Lambda^2 k^n \otimes N \to k^n \otimes N \to N \to 0$$

$$\partial(\underset{\Lambda^i k^n}{\beta} \otimes \underset{N}{\nu}) = \sum_j i_{e_j^*} \beta \otimes t_j \nu).$$

Rem. In fact, for any M the complex $dR(M)$ can be viewed as the Koszul complex of M as a $k[\partial_1, .., \partial_n]$-module, so if M is free over $k[\partial_1, .., \partial_n]$, then $H^i(dR(M)) = 0$ $\forall i \neq 0$ and

$$H^0(dR(M)) = M / (\partial_1, .., \partial_n) M.$$

Now let's pass to $\mathcal{O}$-coherent $\mathcal{D}$-modules, and construct a projective resolution of any such $\mathcal{D}$-module.

Let M, N be any left $\mathcal{D}$-modules. Then $M \otimes_{\mathcal{O}} N$ is also a left $\mathcal{D}$-module, namely

$\nabla(m\otimes n) = \nabla m\otimes n + P^{12}(m\otimes \nabla n)$ (P^{12} permutes components 1 and 2).

i.e. $\partial_i(m\otimes n) = \partial_i m\otimes n + m\otimes \partial_i n$.

Now suppose M is $\mathcal{O}$-coherent. Then $M\otimes_{\mathcal O} dR(\mathcal D)$ is an exact complex except in degree 0, since $M\otimes_{\mathcal O}$ is an exact functor (M is projective). Moreover, $H^0(M\otimes_{\mathcal O} dR(\mathcal D)) = \underline{M}$ (as a ~~right~~ left $\mathcal D$-module).

using $\sigma:\mathcal D\to\mathcal D$

So $M\otimes_{\mathcal O} dR(\mathcal D)$ is a projective resolution of $\underline{M}$. Let us compute $\mathbb D(M)$ using this resolution. Consider the complex $Hom_{\mathcal D}(M\otimes_{\mathcal O} dR(\mathcal D),\mathcal D)$, which looks like:

$$0\to Hom_{\mathcal D}(M\otimes dR^0(\mathcal D),\mathcal D)\to\cdots\to Hom_{\mathcal D}(M\otimes dR^{-n}(\mathcal D),\mathcal D)\to 0.$$

But $Hom_{\mathcal D}(M\otimes_{\mathcal O} dR(\mathcal D),\mathcal D) = M^\vee\otimes_{\mathcal O} dR(\mathcal D)[-n]$.

Thus $\mathbb D(M)=M^\vee$ and Cor 6 is proved.

~~Lecture 22~~

Let us now prove Theorem 2 ~~from last lecture~~. We'll use the following fact from comm. algebra.

<u>Thm 8</u> (Serre) Let R be a regular f.gen. algebra of dimension m and let M be a f.g. R-module. Set $d(M) = \dim(\operatorname{supp} M)$, $j(M) = \min(j \mid \operatorname{Ext}^j(M,R) \neq 0)$. Then

(1) $d(M) + j(M) = m$

(2) $\operatorname{Ext}^j(M,R)$ is a f.g. R-module and $d(\operatorname{Ext}^j(M,R)) \leq m-j$.

(3) For $j = j(M)$ we have an equality in (2).

We will apply this to $R = \operatorname{gr} A$.

<u>Lemma 9</u>. For each filtered A-module M there exists a filtered resolution $P^\bullet$ of M such that $\operatorname{gr} P^\bullet$ is a free resolution of $\operatorname{gr} M$.

<u>Pf.</u> Use homog. generators $\overline{m}_i$, $i \in I_0$, of $\operatorname{gr} M$ (so that $\deg(\overline{m}_i) = k_i$). The map $\beta_0 : R^{I_0} \to M$ is a surjective morphism of graded modules, in which we shift the grading

in the i-th summand by k_i.
Lift $\bar{m}_i$ to $m_i \in M$, $m_i \in F_{k_i} M$.
Then m_i generate M. So we get
a surj map $\alpha_0 : A^{I_0} \to M$, which is also a map of filtered modules, if we shift the filtration as above. By constr. $gr\,\alpha_0 = \beta_0$.
Let $M_1 = \ker \alpha_0$. It is f-gen, (and so is $gr M_1 = \ker \beta_0$) and we can repeat the same construction to get
$\beta_1 : R^{I_1} \twoheadrightarrow_{gr} M_1$, $\alpha_1 : R^{I_1} \twoheadrightarrow M_1$, $gr(\alpha_1) = \beta_1$.
Repeating this process, we obtain the lemma.

Now we'll use the following well known lemma from homological algebra.
Let K° be a filtered complex. Then $gr K^\circ$ is a graded complex.
So we can consider $gr\, H^i(K^\circ)$ and $H^i(gr K^\circ)$.

<u>Lemma</u> 1) $gr\, H^i(K^\circ)$ is a subquotient of $H^i(gr K^\circ)$ (the exact way in which it is so is given by the spectral sequence)

2) ~~If $H^i(grK^\circ) = 0$ for $i > m$ then $gr(H^i(K^\circ)) = H^i(grK^\circ)$~~
~~$H^i(K^\circ) = 0$ for $i \geq m$~~

This implies the theorem together with the previous lemma and the theorem of Serre.

~~Namely, parts (1) and (2) follow~~

Namely ~~by Serre's theorem~~, by lemma 9 and lemma 10 $j(\mathrm{gr}M) \leq j(M)$, and $d(\mathrm{Ext}^j(\mathrm{gr}M, \mathrm{gr}A)) \leq m - j$ while it is $=$ if $j = j(\mathrm{gr}M)$ (by Theorem 8). ~~But~~ So $\mathrm{Ext}^{j(\mathrm{gr}M)}(\mathrm{gr}M, \mathrm{gr}A)$ has strictly largest dimension that other Exts. Hence it survives and remains of the same dimension in the spectral sequence, which implies (1) $-$ (3) of Theorem 2.