Lecture 6

Functional dimension and homological algebra

Let $M$ be a f.g. $D$-module

$d_a(M) = d(M) = \dim \aturia_a(M)$ and $d_g(M) = \dim \aturia(M)$

**Theorem 1** $d_a(M) = d_g(M)$.

To prove this theorem, let’s develop some homological technology.

**Theorem 2** Let $A$ be a filtered algebra over $k$ such that $\gr A$ is a finitely generated commutative regular algebra (i.e., $\gr A = \mathcal{O}(X)$ where $X$ is a smooth alg. variety).

Let $M$ be a f.g. $A$-module, let $d(M) = \dim (\supp \gr F M)$, where $F$ is a good filtration on $M$, and $j(M) = \min \{ j | \Ext^j(M,A) \neq 0 \}$.

Then

1. $d(M) + j(M) = m$
2. $\Ext^d(M, A)$ is a finitely generated right $A$-module, and $d(\Ext^d(M, A)) \leq m - j$
3. For $j = j(M)$, we have an equality in (2).

Before giving a proof, we derive some corollaries.

**Cor.** $d_a(M) = d_g(M) = 2n - j(M)$ for any f.g. $M$. \( \Rightarrow \) **Th. 1.**

From now on we’ll write $d(M)$ for $d(M) = d_g(M)$. 

Claim 3. For any f.g. $D$-modules $M, N$, $\text{Ext}^j(M, N) = 0$ if $j > n$. Thus, the homological dimension of $D$ is $n$.

Proof. Let $P$ be a free f.g. module and $0 \rightarrow K \rightarrow P \rightarrow M \rightarrow 0$ be a short exact sequence. Then, we have

$$\text{Ext}^j(M, P) \rightarrow \text{Ext}^j(M, N) \rightarrow \text{Ext}^{j+1}(K, N) \rightarrow \text{Ext}^{j+1}(K, P).$$

But $\text{Ext}^j(M, P) = \text{Ext}^j(M, P) = 0$ by Thm 2 (2) and the Bernstein inequality.

Also $K$ is finitely generated, as $D$ is Noetherian, and $\text{Ext}^j = 0$ for $j > 2n$ since $\text{gr } D$ has homological dim $= 2n$.

Thus we get the desired statement by induction in $l = 2n - j + 1$ (starting from 0).
Cor 4 \quad M \text{ is holonomic} \iff \text{Ext}^j(M, \mathcal{D}) \neq 0 \text{ only if } j = n.

\textbf{Proof.} \quad M \text{ is holonomic} \Rightarrow \dim(M) = n \text{ for Bernstein filtration} \Rightarrow j(M) = n \text{ by (1) in Thm 2.}
So we have \text{Ext}^j(M, \mathcal{D}) = 0 \text{ for } j < n
by def of j(M). \quad \text{For } j > n \quad \text{Ext}^j(M, \mathcal{D}) = 0
by Thm 2(2), as m = 2n, so \dim(\text{Ext}^j(M, \mathcal{D})) \leq 2n - j,
and thus the statement holds by the Bernstein inequality. \Box

So the only Ext which has a chance of being nonzero is \text{Ext}^n(M, \mathcal{D}). \quad \text{This is a right } \mathcal{D} \text{-module, but we can turn it into a left } \mathcal{D} \text{-module by using the antiautomorphism } 6(x_i) = x_i, \quad 6(\partial_i) = -\partial_i.

\textbf{Cor 5} \quad \text{let } \mathcal{D}(M) = \text{Ext}^n(M, \mathcal{D}) \text{ considered as a left module. Then the functor } 
M \mapsto \mathcal{D}(M) \text{ is an exact contravariant functor on the category of holonomic } \mathcal{D} \text{-modules, and } \mathcal{D}^2 \cong \text{id.}
Proof. Exactness of $\mathcal{D}$ follows immediately from the long exact sequence of $\text{Exts.}$ Let us show that $\mathcal{D}^2 = \text{id}$. Let $P$ be a f.g. projective $\mathcal{D}$-module. Let $P^\vee = \text{Hom}(P, \mathcal{D})$ be the dual module (considered as a left $\mathcal{D}$-module as before). Clearly $P^\vee$ is projective. If $P^n$ is a complex of projective $\mathcal{D}$-modules, then denote by $(P^\vee)_i$ the complex defined by $(P^\vee)_i = (P_{i+n})^\vee$.

Let $M$ be a holonomic $\mathcal{D}$-module and let $P^n$ be its projective resolution. Then, it's clear that $(P^\vee)_i[n]$ (shift by $n$) is a projective resolution of $\mathcal{D}(M)$. This implies that $\mathcal{D}^2 = \text{id}$.

Cor. 6. Let $M$ be $\mathcal{D}$-coherent. Then $\mathcal{D}(M) = \text{Hom}(M, \mathcal{D}) = M^\vee$ as an $\mathcal{D}$-module (i.e. $\mathcal{D}(M)$ is the dual vector bundle).

The dual connection is described as follows: it's the only connection making the pairing between $M$ and $M^\vee$...
Invariant

If \( m \in M \) and \( m' \in M' \), then
\[
(\nabla m, m') + (m, \nabla m') = d(m, m').
\]
So if \( \nabla M = d + \omega \) then \( \nabla_{M'} = d - \omega' \).

Before proving this corollary, let us develop some technology.

**Def.** Let \( M \) be any \( D \)-module.
The De Rham complex of \( M \) is the complex
\[
0 \to M \to \Omega^1 \otimes M \to \cdots \to \Omega^n \otimes M \to 0
\]
(the second row shows the cohomological degrees). The differentials
\[
d : \Omega^i \otimes M \to \Omega^{i+1} \otimes M
\]
are defined by the formula
\[
d (\omega \otimes m) = dw \otimes m + (-1)^i \omega \bigwedge^i \nabla m.
\]
(Leibniz rule). It's easy to check that it is well defined, and \( d^2 = 0 \).
Moreover, if \( M = 0 \), this coincides with the usual De Rham complex.
(if \( M \) is a vector bundle, this is known as the De Rham complex with twisted
coefficients). If \( \nabla = \partial + \omega \), and \( d\omega + [\omega, \omega] = 0 \), then the differential \( d\omega = \partial + c_\omega \) satisfies \( d^2 = 0 \). This is exactly the differential in the above complex.

**Notation.** This complex denoted by \( dR(D) \).

In particular, we can consider the complex \( dR(D) \), which is a complex of free right \( D \)-modules.

**Lemma 7.** \( H^i(dR(D)) = \begin{cases} \mathbb{C}, & i \neq 0 \\ \mathbb{R}^n, & i = 0 \end{cases} \)

(here \( \mathbb{R}^n \) is a right \( D \)-module via \( \lambda D = (\lambda^*)^\vee D \), where \( \lambda^* \) is the adjoint of \( \lambda \) under the integration pairing: \( \partial_i^* = -\partial_i \), \( x_i^* = x_i \) (i.e. \( \lambda^* = \sigma(\lambda) \)).

**Proof.** The complex \( dR(D) \) looks like:

\[ 0 \to D \to \mathbb{R}^n \otimes D \to \Lambda^2 \mathbb{R}^n \otimes D \to \cdots \to \Lambda^n \mathbb{R}^n \otimes D \to 0 \]

with differential given by

\[ d(\lambda \otimes L) = [\partial_i \lambda \otimes dx_i \otimes \partial_i L]. \quad (\lambda \in \Lambda_n \mathbb{R}^n, L \in D) \]

So this is the Koszul complex of \( D \) as a \( \mathbb{R}[\partial_1, \ldots, \partial_n] \)-module (but it's not written down explicitly).
Thus, since \( D \) is a free module over \( k[\partial_1, \ldots, \partial_n] \), we get that the cohomology is nontrivial only in degree 0, and is \( \Lambda^n k^n \otimes (\partial_1, \ldots, \partial_n) D \). \( \square \)

(Recall that the Koszul complex of a module \( N \) over \( k[t_1, \ldots, t_n] \) is \( 0 \rightarrow \Lambda^n k^n \otimes N \rightarrow \cdots \rightarrow \Lambda^1 k^1 \otimes N \rightarrow k^n \otimes N \rightarrow N \rightarrow 0 \),

\[ \partial(\beta \otimes x) = \sum \left( \epsilon_j \beta \otimes \frac{d_j}{d_j} x \right). \]

In fact, for any \( M \) the complex \( dR(M) \) can be viewed as the Koszul complex of \( M \) as a \( k[\partial_1, \ldots, \partial_n] \)-module, so if \( M \) is free over \( k[\partial_1, \ldots, \partial_n] \) then \( H_i(\text{dR}(M)) = 0 \) \( \forall i \neq 0 \) and \( H_0(\text{dR}(M)) = \frac{\text{M}}{(\partial_1, \ldots, \partial_n) \text{M}} \).

Now let's pass to \( O \)-coherent \( D \)-modules and construct a projective resolution of any such \( D \)-module. Let \( M, N \) be any left \( D \)-modules. Then \( M \otimes N \) is also a left \( D \)-module, namely...
\[ D(m \otimes n) = Dm \otimes n + \overset{12}{P} (m \otimes Dn) \quad (P^{12} \text{ permutes components}) \]

i.e. \[ \partial_i (m \otimes n) = \partial_i m \otimes n + m \otimes \partial_i n, \quad 1 \text{ and } 2. \]

Now suppose \( M \) is \( \mathcal{O} \)-coherent. Then \( M \otimes \mathcal{D}(\mathcal{D}) \) is an exact complex except in degree 0, since \( M \otimes \mathcal{D} \) is an exact functor (\( M \) is projective). Moreover,

\[ H^0 (M \otimes \mathcal{D}(\mathcal{D})) = \text{projlim} \mathcal{F} M \text{ (as a left \( \mathcal{D} \)-module).} \]

So \( M \otimes \mathcal{D}(\mathcal{D}) \) is a projective resolution of \( M \). Let us compute \( \mathcal{D}(M) \) using this resolution. Consider the complex \( \text{Hom}_\mathcal{D} (M \otimes \mathcal{D}(\mathcal{D}), \mathcal{D}) \), which looks like:

\[
0 \rightarrow \text{Hom}_\mathcal{D} (M \otimes \mathcal{D}(\mathcal{D}^0), \mathcal{D}) \rightarrow \cdots \rightarrow \text{Hom}_\mathcal{D} (M \otimes \mathcal{D}(\mathcal{D}^0), \mathcal{D}) \rightarrow 0.
\]

But \[ \text{Hom}_\mathcal{D} (M \otimes \mathcal{D}(\mathcal{D}), \mathcal{D}) = M \otimes \mathcal{D}(\mathcal{D})[\mathcal{O}] \]

Thus \( \mathcal{D}(M) = M \otimes \mathcal{D}(\mathcal{D})[\mathcal{O}] \) and Cor 6 is proved.
Let us now prove Theorem 2 from last lecture. We'll use the following fact from comm. algebra.

**Thm 8 (Serre)** Let $R$ be a regular algebra of dimension $M$ and let $M$ be a f.g. $R$-module. Set $d(M) = \dim(\text{supp} M)$, $j(M) = \min \{ j \mid \text{Ext}^j(M, R) \neq 0 \}$. Then

1. $d(M) + j(M) = M$
2. $\text{Ext}^d(M, R)$ is a f.g. $R$-module and $d(\text{Ext}^d(M, R)) \leq m - j$.
3. For $j = j(M)$ we have an equality in (2).

We will apply this to $R = \text{gr}A$.

**Lemma 9.** For each filtered $A$-module $M$ there exists a filtered resolution $P^\bullet$ of $M$ such that $\text{gr} P^\bullet$ is a free resolution of $\text{gr} M$.

**Pf.** Use homog. generators $\bar{m}_i$, of gr $M$ (so that $\deg(\bar{m}_i) = k_i$). The map $\beta : R \rightarrow M$ is a surjective morphism of graded modules, in which we shift the grading
in the $i$-th summand by $k_i$.

lift $m_i$ to $m_i \in M'$, $m_i \in F_k M$.

Then $m_i$ generate $M$. So we get a map $A^i \to M$, which is also a map of filtered modules if we shift the filtration as above. By constr. $\text{gr}x_0 = \beta_0$ (and so is $\text{gr}M = \ker \beta_0$).

Let $M' = \ker \alpha_0$. It is $\beta_0$-acyclic, and we can repeat the same construction to get $\beta_1: R^1 \to \text{gr}M$, $\alpha_1: R^1 \to M$, $\text{gr}(\alpha_1) = \beta_1$.

Repeating this process, we obtain the lemma.

Now we'll use the following well-known lemma from homological algebra.

Let $K^\circ$ be a filtered complex. Then $\text{gr} K^\circ$ is a graded complex.

So we can consider $\text{gr} H^i(K)$ and $H^i(\text{gr} K^\circ)$.

Lemma: $\text{gr} H^i(K^\circ)$ is a subquotient of $H^i(\text{gr} K^\circ)$ (the exact way in which it is so is given by the spectral sequence).
This implies the theorem together with the previous lemma and the theorem of Serre.

Namely, parts 1 and 2 follow. Namely by Serre's theorem, by Lemma 10 $j(\text{gr}M) \leq j'(M)$, and $d(\text{Ext}^i(\text{gr}M, \text{gr}A)) \leq m - j$ while it is $= i$ if $j = j(\text{gr}M)$ (by Theorem 8). But so $\text{Ext}^i(\text{gr}M)(\text{gr}M, \text{gr}A)$ has largest dimension that other $\text{Ext}^i$s. Hence it survives and remains of the same dimension in the spectral sequence, which implies (1) - (3) of Theorem 2.