Lectures 4,5

<u>$\mathcal{O}$-coherent $\mathcal{D}$-modules.</u> Let $\mathcal{O} = k[x_1, \dots, x_n] \subset \mathcal{D}$. Let M be a f.g. $\mathcal{D}$-module. Then $SS(M) \subseteq \{\xi_1 = \dots = \xi_n = 0\}$ $\iff$ M is f.g. over $\mathcal{O}$. This is equivalent to saying that on $\mathrm{gr} M$, ξ_i act nilpotently. Such $\mathcal{D}$-modules will be called $\mathcal{O}$-coherent.

<u>Theorem 1</u> If M is $\mathcal{O}$-coherent then M is locally free over $\mathcal{O}$ ($\iff$ M is a module of sections of a vector bundle over $\mathbb{A}^n$).

<u>Proof.</u> Let $x \in \mathbb{A}^n$ and $\mathcal{O}_x$ be the local ring of x. Let $m_x \subset \mathcal{O}_x$ be the max. ideal. Let M_x be the localization of M, $M_x = \mathcal{O}_x \otimes_{\mathcal{O}} M$. Since M is $\mathcal{O}$-coherent, $\dim_k M_x / (m_x \cdot M_x) < \infty$. Let $\bar{s}_1, \dots, \bar{s}_\ell$ be a basis of this space. Lift this basis to elements $s_1, \dots, s_\ell \in M$. By Nakayama's lemma, $s_1, \dots, s_\ell$ generate M_x. We have to show that these elements are linearly independent over $\mathcal{O}_x$. So assume that $\sum_i \varphi_i s_i = 0$, $\varphi_i \in \mathcal{O}_x$, and assume that not all $\varphi_i = 0$.

We say that $\mathrm{ord}_x \varphi = n$ if $\varphi \in m_x^n$, but $\varphi \notin m_x^{n+1}$. Let $\nu = \min \mathrm{ord}\, \varphi_i$. WLOG can assume that ($\nu$ is minimal possible, and) $\mathrm{ord}_x \varphi_1 = \nu$. (clearly $\nu \geq 0$). Then $\exists$ a vector field η (defined locally around x) such that $\eta(\varphi_1) \neq 0$ and $\mathrm{ord}_x \eta(\varphi_1) < \nu$.

Let's apply such an η to the expression $\sum_i \varphi_i s_i = 0$. We get

$$0 = \sum_i \eta(\varphi_i)\, s_i + \sum_i \varphi_i\, \eta(s_i).$$

Since s_i generate M_x, we have

$$\eta(s_i) = \sum_j a_{ij} s_j, \quad a_{ij} \in \mathcal{O}_x.$$

So

$$0 = \sum_i \Big(\eta(\varphi_i) + \sum_j \varphi_j a_{ij}\Big) s_i .$$

The coeff. of s_1 is $\eta(\varphi_1) + \sum_j \varphi_j a_{j1}$

So this coefficient is nonzero and has order $< \nu$ at x. This is a contradiction since we assumed that ν is minimal possible. Thus $\varphi_i = 0$ for all i $\blacksquare$

<u>Cor 2</u> M is $\mathcal{O}$-coherent $\Leftrightarrow$ $SS(M) = \{(x,0) \mid x \in \mathbb{A}^n\}$, the 0-section of $T^*\mathbb{A}^n$.

Pf. Let M be $\mathcal{O}$-coherent. Take a filtration on M defined by $F_i M = M$ (for the order filtration on $\mathcal{D}$). Then F is good, as $\mathrm{gr} M = M$ as an $\mathcal{O}$-module. So $SS(M) = \{(x,0) \mid x \in \mathbb{A}^n\}$. Conversely, if $SS(M) = \{(x,0) \mid x \in \mathbb{A}^n\}$ then $\mathrm{gr} M$ is finitely generated over $\mathbb{C}[x_1, \ldots, x_n]$, so M is also finitely generated over $\mathbb{C}[x_1, \ldots, x_n]$. $\blacksquare$.

Ex. If $M = \delta_a$ then $SS(M) = \{x = a\}$, so M is not $\mathcal{O}$-coherent. (note that $SS_a(M) = \{x = 0\}$.

Flat connections. One can think of $\mathcal{D}$-modules as $\mathcal{O}$-modules with an additional structure. This additional structure turns out to be a flat connection. Namely, any $\mathcal{D}$-module M comes with a map $\nabla : M \to \Omega^1 \otimes_{\mathcal{O}} M$, where Ω^1 is the module of 1-forms defined as follows: If v is a vector field then $\nabla_v = (\nabla, v \otimes 1) : M \to M$ is $\nabla_v(m) = v(m)$. This map satisfies the following properties: $\nabla(fm) = df \otimes m + f\nabla(m)$, $\forall f \in \mathcal{O}, m \in M$. Such a map ∇ is called a connection on an $\mathcal{O}$-module.

Given a connection ∇, we can define a map $\nabla^2 : M \longrightarrow \Omega^2(\mathbb{A}^n) \otimes_{\mathcal{O}} M$ ($\mathcal{O}$-linear) in the following way:

$$\nabla^2(m) = (1 \otimes \nabla)(\nabla(m)) - (d \otimes 1)\nabla(m),$$

where d is the De Rham differential. The map ∇^2 is called the curvature of ∇.

Checking that ∇^2 is well defined: Suppose we replace $\nabla(m)$ with $\nabla(m) + \alpha \otimes f m' - f\alpha \otimes m'$. Then $(1 \otimes \nabla)(\nabla(m))$ will change by $\alpha \otimes \nabla(fm') - f\alpha \otimes \nabla(m') \; (= \alpha \wedge df \otimes m')$, and $(d \otimes 1)\nabla(m)$ will change by $d\alpha \otimes fm' - d(f\alpha) \otimes m'$
$= -df \wedge \alpha \otimes m' = \alpha \wedge df \otimes m'$ (note that $(1\otimes\nabla)(\nabla(m))$ and $(d \otimes 1)\nabla(m)$ are defined only on $\Omega^1 \otimes_{\mathbb{C}} M$).

Checking that ∇^2 is linear:

$$\nabla^2(gm) = (1 \otimes \nabla)(\nabla(gm)) - (d \otimes 1)(\nabla(gm))$$

$$= (1 \otimes \nabla)(g\nabla(m) + dg \otimes m) - (d \otimes 1)(g\nabla(m) + dg \otimes m)$$

$$= g(1 \otimes \nabla)(\nabla(m)) + \cancel{dg \otimes \nabla(m)} - \cancel{dg \otimes \nabla(m)} - g(d \otimes 1)(\nabla(m)).$$

Exer. $\nabla^2(m)(v,w) = \nabla_v \nabla_w m - \nabla_w \nabla_v m - \nabla_{[v,w]} m$, $v, w \in$ Vector fields.

Def. A connection ∇ is called flat if $\nabla^2 = 0$.

Prop 2. A $\mathcal{D}$-module on $\mathbb{A}^n$ is the same thing as an $\mathcal{O}$-module with a flat connection

Pf. Given an $\mathcal{O}$-module M with a connection ∇, we can set $\partial_i = \nabla_{\partial_i} : M \to M$, and $[\partial_i, \partial_j] = 0 \iff \nabla$ is flat. Also ∇_{∂_i} define the connection uniquely.

Rem. Proof of thm. that an $\mathcal{O}$-coherent $\mathcal{D}$-module is locally free is based on the fact that we have a connection so "dimension of fibers should be the same (we can carry vectors from one fiber to another). Think why this kind of intuition fails for "∞-dimensional" case ($M/\mathcal{O}$ inf. generated).

On $\mathbb{A}^n$, any vector bundle is trivial (Quillen thm), so an $\mathcal{O}$-coherent $\mathcal{D}$-module on $\mathbb{A}^n$ has the form $\mathcal{O}\otimes V$, and a connection has the form $\nabla = d + \omega$, where $\omega \in \Omega^1(\mathbb{A}^n)\otimes \operatorname{End} V$.

Exercise. ∇ is flat $\iff d\omega + \frac{1}{2}[\omega, \omega] = 0$. (the Maurer-Cartan equation)

Poisson structures. Recall that if R is a commutative algebra then a Poisson bracket on R is a Lie bracket satisfying the following condition:

(*) $\{f_1 f_2, g\} = f_1\{f_2, g\} + \{f_1, g\} f_2 \quad \forall f_1, f_2, g \in R$.

Let X be a smooth affine alg. variety $/k$. Any skewsymmetric bilinear map $\{,\} : \mathcal{O}_X \otimes \mathcal{O}_X \to k$

satisfying (*) gives rise to a bivector field $\eta \in \Lambda^2 T_X$. Any such η defines a skewsymmetric map $\eta : T_X^* \to T_X$. If η is an isomorphism, then $\eta^{-1} : T_X \to T_X^*$ corresponds to a 2-form $\omega \in \Lambda^2 T_X^*$. It's well known that $\{,\}$ satisfies the Jacobi identity if and only if ω is closed: $d\omega = 0$. In this case ω is called a <u>symplectic form.</u> I.e.

<u>Def</u>. A closed nondegenerate 2-form ω on X is called a symplectic form.

<u>Ex</u>. Suppose Y is a smooth variety. Then T^*Y carries a canonical 1-form α: $\forall v \in T_{(x,\xi)} T^*Y$, $\alpha(v) = \langle \xi, \bar{v} \rangle$, where $\bar{v}$ is the projection of v to TY under $d\pi : TT^*Y \to TY$, where $\pi : T^*Y \to Y$. If x_i are local coordinates on Y and $\xi_i = dx_i$ the corresponding elements (coordinates) of T^*Y, then x_i, ξ_i are local coordinates on T^*Y, and $\eta = \sum \xi_i dx_i$. So $\omega = d\eta = \sum d\xi_i \wedge dx_i$, hence ω is a canonical symplectic form on $X = T^*Y$. The Poisson bracket attached to this form in local coordinates

looks like
$$\{F,G\} = \sum\left(\frac{\partial F}{\partial \xi_i}\frac{\partial G}{\partial x_i} - \frac{\partial F}{\partial x_i}\frac{\partial G}{\partial \xi_i}\right).$$
So $\{\xi_i, x_j\} = \delta_{ij}$, and other brackets of ξ_i and x_i are zero.

Assume now that A is a filtered algebra, and there exists $\ell > 0$ such that $[F_iA, F_jA] \subseteq F_{i+j-\ell}A$. Then $\mathrm{gr}A$ is commutative and endowed with a canonical Poisson bracket of degree $-\ell$. Namely, for $\bar{x} \in \mathrm{gr}_iA$ and $\bar{y} \in \mathrm{gr}_jA$, let $x \in F_iA$ and $y \in F_jA$ be any lifts, and set $\{\bar{x}, \bar{y}\}$ to be the image of $[x,y]$ in $\mathrm{gr}_{i+j-\ell}A$. It's easy to show that this is well defined and is a Poisson bracket.

Ex. If $A = \mathcal{D}$ with Bernstein filtration, $\ell = 2$, or $A = \mathcal{D}$ with order filtration, $\ell = 1$, then the bracket we get is the standard bracket on $\mathbb{A}^{2n}$ discussed above...

Let X be a Poisson affine algebraic variety (i.e. $\mathcal{O}_X$ is a Poisson algebra) and $Z \subset X$ a closed subvariety.

<u>Def.</u> Z is called coisotropic if the defining ideal $I(Z)$ of Z is a Poisson subalgebra of $\mathcal{O}_X$, i.e. $\{I(Z), I(Z)\} \subset I(Z)$.

Suppose X is ~~symplectic~~ smooth. We have $T_z Z \subset T_z X$ and $T_z Z^\perp \subset T_z^* X$ (say, for smooth points $z \in Z$).

<u>Lemma 4.</u> Z is coisotropic $\Leftrightarrow$ $\forall z \in Z$ (a smooth point), the Poisson bivector $\eta \in \Lambda^2 T_X$ vanishes as a bilinear form on $T_z Z^\perp$, i.e. $\eta(T_z Z^\perp) \subset T_z Z$.

<u>Proof.</u> Exercise.

<u>Corollary 5.</u> If X is symplectic and $Z \subset X$ coisotropic then $\forall$ irreducible component of Z has dimension $\geq \frac{1}{2} \dim X$.

A central role in the theory of D-modules is played by the following theorem, due to Gabber.

Let A be a filtered algebra, and $[F_i A, F_j A] \leq F_{i+j-\ell} A$. Suppose $\mathrm{gr} A$ is finitely generated commutative.

Then it's easy to see that if M is an A-module with a good filtration and $I = \mathrm{Ann}(\mathrm{gr} M)$ then $\{I, I\} \subset I$ (for $\ell=1$). However, I depends on the filtration, so it is more interesting to look at $J = J(M) = \sqrt{I}$, which does not depend on the filtration.

<u>Theorem 5</u> (Gabber) $\{J, J\} \subset J$.

This theorem is nontrivial, and a proof for $\ell=1$ is given in the appendix (proof for $\ell>1$ is similar)

<u>Corollary 6</u>. Let M be a f.g. $\mathcal{D}$-module on $\mathbb{A}^n$. Then $SS(M)$ and $SS_a(M)$ are coisotropic, so every irreducible component of these varieties has dimension $\geq n$. (this is a geometric version of the Bernstein inequality, and it implies the Bernstein inequality).

More generally, if Y is a smooth affine variety, then similarly to the case $Y = \mathbb{A}^n$, we can define, for any f.g. $\mathcal{D}$-module M over $\mathcal{D}(Y)$, a subvariety $SS(M) \subset \mathcal{O}(T^*Y) = \mathrm{gr}\, \mathcal{D}(Y)$, and $SS(M)$ is coisotropic. We'll see it later.

 Let char $k=0$.

Gabber's theorem. Let $\bar{A}$ be a commutative finitely generated k-algebra (in applications ~~usually~~ often $\bar{A}=k[y_1,..,y_n]$). Let $\bar{M}$ be a f.g. $\bar{A}$-module. Let A, M be flat deformations of $\bar{A}, \bar{M}$ over $k[\varepsilon]/\varepsilon^2$. For $\bar{a}_1, \bar{a}_2 \in \bar{A}$, let $\{\bar{a}_1, \bar{a}_2\} \in \bar{A}$ be defined by the condition that $[a_1, a_2] = \varepsilon\{\bar{a}_1, \bar{a}_2\}$ for any lifts a_1, a_2 of $\bar{a}_1, \bar{a}_2$. Let $I=\sqrt{\mathrm{Ann}_{\bar{A}}\bar{M}}$.

Theorem 1 (Gabber). One has $\{I, I\} \subseteq I$.

Proof. Let Z be an irreducible component of the zero set $Z(I) \subset \mathrm{Spec}\,\bar{A}$, and $\mathfrak{p} \subset \bar{A}$ be the corresponding prime ideal. The ideal I is the intersection of finitely many such primes corresponding to components of $Z(I)$), so it suffices to check that $\{\mathfrak{p}, \mathfrak{p}\} \subset \mathfrak{p}$.

Now, $\{,\}$ is a biderivation of $\bar{A}$, so $\{\mathfrak{p}, \mathfrak{p}\} \subset \mathfrak{p}$ is a local condition, and can be checked after replacing A by $A_f \overset{\text{def}}{=} A[f^{-1}]$, where $f \in A$ is such that $\bar{f} \in \bar{A}/\mathfrak{p}$ (image of f) is ~~a~~ nonzero ~~divisor, not contained in $\mathfrak{p}$ (i.e. $\bar{f}|_Z \neq 0$).~~

Pick $x_1, .., x_\ell \in A$ such that $\bar{x}_i$ define a maximal set of algebraically independent elements in $\bar{A}/\mathfrak{p}$. Let $R \subset \bar{A}$ be $R=k[\bar{x}_1, .., \bar{x}_\ell]$. We'll pick $\bar{f} \in R$ in such a way that

$\bar{f}=\bar{x}_1 \cdot \bar{f}'$

and $\bar{x}_1$ vanishes on all components of $Z(I)$ except

1) $\wp^s \overline{M}_f = 0$ for some $s \geq 0$ (where M_f is the localization of M with respect to f). This ~~means~~ is true ~~that~~ all components of $Z(I)$ other that Z are contained in $\bar{f} = 0$ (but $\bar{f}|_Z \neq 0$).

2) $B \overset{\text{def}}{=} \overline{A}_f / \wp_f$ is a free R_f-module (here subscript f means localization w.r.t. f).

3) $\wp^i \overline{M}_f / \wp^{i+1} \overline{M}_f$ is a free B-module (hence a free R_f-module) for all i.

It's easy to see that such f exists.

Now choose an R_f-basis $\overline{m}_1, \dots, \overline{m}_{N_1}$ of $\overline{M}_f / \wp \overline{M}_f$, $\overline{m}_{N_1+1}, \dots, \overline{m}_{N_2}$ of $\wp \overline{M}_f / \wp^2 \overline{M}_f$, and so on. We have

$$\wp_f \overline{m}_i \subseteq \sum_{j>i} R_f \overline{m}_j \in \mathrm{gr}\, \overline{M}_f .$$

(here $j > i$ means that $\overline{m}_j \in \wp^{j'} \overline{M}_f / \wp^{j'+1} \overline{M}_f$, $\overline{m}_i \in \wp^{i'} \overline{M}_f / \wp^{i'+1} \overline{M}_f$ and $j' > i'$).

Let m_i be lifts of $\overline{m}_i$ to $\overline{M}_f$.

<u>Lemma</u>. Let $\bar{a}, \bar{b} \in \wp_f$. Then there exist $e_{ij} \in R_f$ such that $\{\bar{a}, \bar{b}\} m_i = \sum_j e_{ij} m_j$, and $\sum_i e_{ii} = 0$.

Let us show how the lemma implies the theorem, and then prove the lemma.

<u>Lemma $\Rightarrow$ Theorem.</u> The lemma implies that $\mathrm{tr}_{R_f}(\{\bar{a}, \bar{b}\}, \overline{M}_f) = 0$. In this identity, we can replace $\bar{a}$ by $x\bar{a}$ for any $x \in \overline{A}_f$ (as $x\bar{a} \in \wp_f$).

So $0 = tr_{R_f}(\{x\bar{a},\bar{b}\},\bar{M}_f) = tr_{R_f}(x\{\bar{a},\bar{b}\} + \bar{a}\{x,\bar{b}\},\bar{M}_f)$.

Now, since $\bar{a}\in\wp$, multiplication by $\bar{a}\{x,\bar{b}\}$ strictly preserves the filtration on $\bar{M}_f$ (by $\wp^i\bar{M}_f$), so the trace of the second term is zero, and we get that $tr_{R_f}(x\{\bar{a},\bar{b}\},\bar{M}_f)=0$.

~~But M_f is a free B-module~~

Now, $x\{\bar{a},\bar{b}\}$ preserves the filtration, so we get $0=tr_{R_f}(x\{\bar{a},\bar{b}\}, gr\bar{M}_f)=$

$$= \sum_i tr_{R_f}(x\{\bar{a},\bar{b}\}, \wp^i\bar{M}_f/\wp^{i+1}\bar{M}_f).$$

But $\wp^i\bar{M}_f/\wp^{i+1}\bar{M}_f$ are free B-modules of some ranks r_i. So we get

$$0=\underbrace{\left(\sum_i r_i\right)}_{\neq 0 \text{ in } \mathbb{Z}} tr_{R_f}(x\{\bar{a},\bar{b}\},B). \qquad \forall x\in B.$$

Thus, as char $k=0$, we get $tr_{R_f}(x\{\bar{a},\bar{b}\},B)=0$.

But generically over $\operatorname{Spec} R_f$, B is a semi-simple algebra, so if $tr_{R_f}(x\{\bar{a},\bar{b}\},B)=0$ $\forall x\in B$ then $\{\bar{a},\bar{b}\}=0$ in B as desired. (we used Frobenius property of B).
($\Rightarrow \{a,b\}\in\wp_f$ as element of $\bar{A}_f$).

It remains to prove the lemma.

<u>Proof of the Lemma.</u>

Let $A'\subseteq A_f$ be the set of z such that $\bar{z}\in R_f$.
Let $a,b\in A_f$ be any lifts of $\bar{a},\bar{b}$.
Let $\hat{m}_i$ be any lifts of m_i to M_f.

We have (since $\bar{a} \in \mathcal{P}$):

$$a\hat{m}_i = \sum_{j \geq i} u^0_{ij} \hat{m}_j + \varepsilon \sum_j u^1_{ij} \hat{m}_j,$$

$u^0_{ij} \in A'$, $u^1_{ij} \in R_f$. Similarly

$$b\hat{m}_i = \sum_{j \geq i} v^0_{ij} \hat{m}_j + \varepsilon \sum_j v^1_{ij} \hat{m}_j$$

$v^0_{ij} \in A'$, $v^1_{ij} \in R_f$.

(indeed, we have $\bar{a} m_i = \sum_{j \geq i} \bar{u}^0_{ij} m_j$, $\bar{b} m_i = \sum_{j \geq i} \bar{v}^0_{ij} m_j$, $\bar{u}^0_{ij}, \bar{v}^0_{ij} \in R_f$,
and we can pick u^0_{ij}, v^0_{ij} to be any lifts of $\bar{u}^0_{ij}$, $\bar{v}^0_{ij}$; then $a\hat{m}_i - \sum_{j \geq i} u^0_{ij} m_j$ is a multiple of ε, so can be written as $\varepsilon \sum_j u^1_{ij} \hat{m}_j$ $(\in \varepsilon \bar{M}_f)$, $u^1_{ij} \in R_f$, and similarly for $b\hat{m}_i$).

Thus, denoting by U^0, U^1, V^0, V^1 the matrices formed by $u^0_{ij}, u^1_{ij}, v^0_{ij}, v^1_{ij}$, we get

$$ab\hat{m}_i = \sum_{j \geq i} \left(v^0_{ij} a\hat{m}_j + \varepsilon \{\bar{a}, v^0_{ij}\} \hat{m}_j \right) + \varepsilon \sum_j v^1_{ij} a\hat{m}_j$$

$$= \sum_{j \geq i} \left((V^0 U^0)_{ij} \hat{m}_j + \varepsilon \{\bar{a}, v^0_{ij}\} \hat{m}_j \right) +$$

$$+ \varepsilon \sum_j (V^0 U^1 + V^1 U^0)_{ij} \hat{m}_j.$$

There is a similar formula for $ba\hat{m}_i$.

Subtracting them, we get

$$[a,b]\hat{m}_i = \sum_{j\succ i} c^0_{ij}\hat{m}_j + \varepsilon \sum_j c^1_{ij}\hat{m}_j,$$

where $(c^0_{ij}) = C^0$, $(c^1_{ij}) = C^1$ and

$$C^0 = [V^0, U^0] + \varepsilon(\{\bar{a}, V^0\} - \{\bar{b}, U^0\}),$$

and

$$C^1 = [V^0, U^1] + [V^1, U^0].$$

Now, since $[\bar{a}, \bar{b}] = 0$, we have $[V^0, U^0] = 0$ modulo ε. So $C^0_{ij} \in \varepsilon A$. So

$\sum_{j\succ i} c^0_{ij}\hat{m}_j = \varepsilon \sum_{j\succ i} d_{ij}\hat{m}_j$, where $d_{ij} \in R_f$ (we rewrite the A-action via R_f).

Now we can set $e_{ij} = d_{ij} + c^1_{ij}$. Then $\{\bar{a}, \bar{b}\} m_i = \sum_j e_{ij} m_j$

As $d_{ii} = 0$, we have $\sum e_{ii} = \sum c^1_{ii} = \mathrm{Tr}\, C^1$, and $\mathrm{Tr}\, C^1 = 0$ as C^1 is a sum of two commutators. The lemma is proved.

Corollary 2 Let A be a filtered algebra such that $\bar{A} = \mathrm{gr}A$ is commutative and finitely generated, and let M be an ~~filtered~~ A-module with a good filtration. Let $I = \sqrt{\mathrm{Ann}_{\bar{A}} \bar{M}}$, and let $\{,\}$ be the Poisson bracket on $\bar{A}$ induced by A. Then $\{I, I\} \subseteq I$.

Proof. Let $\mathrm{Rees}(A) = \bigoplus_i F_i A \cdot \varepsilon^i$, $\mathrm{Rees}\, M = \bigoplus_i F_i M \cdot \varepsilon^i$ and $\tilde{A} = \mathrm{Rees}(A)/\varepsilon^2 \mathrm{Rees}(A)$, $\tilde{M} = \mathrm{Rees}\, M/\varepsilon^2 \mathrm{Rees}\, M$. Apply ~~this to~~

Generalization: $\bar{A}$ commutative finitely generated algebra, $\bar{M}$ f.g. $\bar{A}$-modules A, M flat deformations of $\bar{A}$ $\bar{M}$ over $k[\varepsilon]/\varepsilon^{l+1}$ s.that A is trivialised modulo ε^l. For $\bar{a}_1, \bar{a}_2 \in \bar{A}$ let $\{\bar{a}_1, \bar{a}_2\} \in \bar{A}$ be defined by $[a_1, a_2] = \varepsilon^l \{\bar{a}_1, \bar{a}_2\}$ for any lifts a_1, a_2 of $\bar{a}_1, \bar{a}_2$. Let $I = \sqrt{Ann_{\bar{A}} \bar{M}}$.

Thm 3 One has $\{I, I\} \subseteq I$.

The proof is the same.

Corollary 4 Let A be a filtered algebra with $F_i A \cdot F_j A \subset F_{i+j-l} A$, s.t. $\bar{A} = grA$ is commutative f.g., and let M be an A-module with a good filtration. Let $I = \sqrt{Ann_{\bar{A}} \bar{M}}$, and $\{,\}$ be the Poisson bracket on $\bar{A}$ of degree $-l$ induced by A. Then $\{I, I\} \subseteq I$.

Pf. Let $\tilde{A} = Rees(A)/\varepsilon^{l+1} Rees(A)$
$\tilde{M} = Rees(M)/\varepsilon^{l+1} Rees(M)$. Apply Thm to $\tilde{A}, \tilde{M}$.