Lecture 3.

Let us study some further properties of the algebra $\mathcal{D}$.

Lemma 1. $\mathcal{D}$ is left & right noetherian.

Pf. Let M be a f.g. $\mathcal{D}$-module, and $N \subset M$. Let F be a good filtration on M. Then $\mathrm{gr}M$ is a f.g. module over polynomials, and $\mathrm{gr}N \subset \mathrm{gr}M$. Hence by the Hilbert basis thm, $\mathrm{gr}N$ is f.g., hence so is N. $\blacksquare$

Let us explain that $\mathcal{D}$-modules are algebraic objects which correspond to systems of linear PDEs.

Suppose we have a system of linear PDEs:

(1) $\sum_j L_{ij} f_j = 0, \qquad i=1,\dots,m, \quad j=1,\dots,r$

where L_{ij} are linear differential operators. Let $Y_{\mathcal{S}}$ be the space of solutions of (1) in some class of functions $\mathcal{S}$, e.g. $\mathcal{S} = C^\infty(\mathbb{R}^n)$ or $\mathcal{S} = C_0^\infty(\mathbb{R}^n)'$ (distributions). Then $Y_{\mathcal{S}}$ can be described in terms of the theory of $\mathcal{D}$-modules. Namely

Let $M = \mathcal{D}^r / \sum_j L_{ij} f_j = 0$ $\quad \mathcal{D}^r = \mathcal{D}f_1 \oplus \cdots \oplus \mathcal{D}f_r$.

Then solutions Y_S of (1) in S are

$$Y_S = \mathrm{Hom}_{\mathcal{D}}(M, S).$$

So in a sense a system of linear PDE in $f_1, \dots, f_r$ is a $\mathcal{D}$-module with a system of generators. Note that the Noetherian property implies that it's always enough to consider finitely many equations.

It is especially interesting to look at the case when M is generated by one element (cyclic $\mathcal{D}$-module). Then $M = \mathcal{D}/I$, where I is a left ideal.

~~We have $I = ($[illegible]$)$~~

It turns out that there are in some sense a lot of cyclic modules.

<u>Proposition 2.</u> Any holonomic $\mathcal{D}$-module M is cyclic.

The proof is based on the following:

Lemma 3. $\mathcal{D}$ is a simple algebra, i.e. it has no proper 2-sided ideals.

Pf. Let $I \overset{\neq 0}{\subset} \mathcal{D}$ be a 2-sided ideal, and $a \overset{\neq 0}{\in} I$. Then $a \in F_i \mathcal{D}$ for some i. Then $[a, x_m]$ or $[a, \partial_m]$ is nonzero and in $F_{i-1}\mathcal{D}$. So $\exists\, a \underset{\in I}{\neq 0}$, $a \in F_0 \mathcal{D} = k$.

Thus $I = \mathcal{D}$.

Proposition 4. Let A be a simple algebra which has infinite length as a left A-module. Then any A-module of finite length is cyclic.

Proof. By induction on the length of M it's enough to show that if we have an exact sequence of A-modules

$$0 \to K \to M \xrightarrow{\pi} N \to 0$$

where $K \neq 0$ is simple and N is cyclic of finite length, then M is also cyclic.

Let $n \in N$ be a generator, $I = \mathrm{Ann}_A(n)$ the annihilator of n in A. Assume that there is no $m \in \pi^{-1}(n)$ generating M. We claim that then I annihilates any element of K.

Indeed, let $m \in \pi^{-1}(n)$, and let $M' = A \cdot m$ $(M' \neq M)$. Then $\pi: M' \to N$ is an isomorphism, since $M' \cap K = 0$ (if $M' \cap K \neq 0$ then $K \subset M'$ since K is simple, and $M = M'$ because of the exactness of the sequence). Thus $Ann_A(m) = I$. So $\forall k \in K$ $Ann(m+k) = I$, so $I \cdot k = 0$ for any $k \in K$.

Thus $I \subset I' = \bigcap_{k \in K} Ann(k)$. I' is a 2-sided ideal, so $I' = 0$, hence $I = 0$. Thus, $N = A$, but it has infinite length. ⊠.

<u>Corollary 5</u>. Any $\mathcal{D}$-module of finite length is cyclic.

This in particular implies Prop. 2.

<u>The singular support of a D-module</u>

If M is a $\mathcal{D}$-module on $\mathbb{A}^n$ with a good filtration (under Bernstein filtration on $\mathcal{D}$), then $gr M$ is a f.g. module over $k[x_1, \dots, x_n, \xi_1, \dots, \xi_n]$, so it has support $supp(gr M)$, a closed subvariety of $\mathbb{A}^{2n}$. It is clear from commutative algebra that $\dim supp(gr M) = d(M)$. Also $supp(gr M)$ is invariant under dilations (as M is a graded module)

Therefore, $S = \mathrm{supp}(\mathrm{gr} M)$ defines a closed subvariety S in $\mathbb{P}^{2n-1}$. In fact, it is more than that: every irreducible component ~~of~~ Z of S comes with a multiplicity c_Z, which is the rank of $\mathrm{gr} M$ at a generic point of Z. It is clear that $c(M) = \sum_{Z : d_m(Z) = d(M)} c_Z \cdot \deg(PZ)$, where for a projective variety $X^d \subset \mathbb{P}^N$, $\deg(X)$ is $d!$ times the leading coefficient of the Hilbert polynomial of X." Thus the sum $\sum_{Z : \dim(Z) = d(M)} c_Z \cdot PZ$ is a cycle of dimension $d(M)$, and degree $c(M)$ in $\mathbb{P}^{2n-1}$ (here PZ is the projectivization of Z). One can also consider $\sum_{\text{all } Z} c_Z PZ$, cycle of mixed dimension.

<u>Lemma 6</u>. In the above notation, let $I_F = \mathrm{Ann}(\mathrm{gr}^F M)$. Then $\sqrt{I_F}$ does not depend on F (i.e., S does not depend on F).

<u>Proof</u>. Let F, F' be two good filtrations on M. They are equivalent: $F'_{j-j_0} M \subset F_j M \subset F'_{j+j_1} M$. Let $t = j_0 + j_1 + 1$. Let $\bar{f} \in k[x_1, \dots, x_n, \xi_1, \dots, \xi_n]$, $\deg(\bar{f}) = p$, $\bar{f} \in \sqrt{I_F}$. Let's lift $\bar{f}$ to $f \in F_p \mathcal{D}$. Since $\bar{f} \in \sqrt{I_F}$, $\exists q$ such that $f^q \cdot F_i M \subset F_{i+pq-1} M$. Then it's easy to check that $f^{qt} F'_i M \subset F'_{i+pq-1} M$. Thus $\bar{f}^{qt} \in I_{F'}$ and thus $\bar{f} \in \sqrt{I_{F'}}$. ∎

In fact, the same argument proves a more general result:

<u>Prop 7.</u> If A is a filtered algebra such that $\mathrm{gr}A$ is commutative ~~and F_iA is a Noetherian module over F_0A~~. Then for a finitely generated module M over A, $\mathrm{supp}(\mathrm{gr}M)$ is canonically defined i.e. for a good filtration F on M, $\sqrt{\mathrm{Ann}\,\mathrm{gr}^F M}$ does not depend on F.

<u>Example.</u> The ideal I_F itself may depend on F. For example, let $M=\mathbb{C}[x]$ for $n=1$ (a module over $\mathcal{D}(\mathbb{A}^1)$). For the usual filtration ($F_iM=\langle 1,\dots,x^i\rangle$), $I_F\subset\mathbb{C}[x,\xi]$ is $I_F=(\xi)$. However, consider the filtration F' on $M=\mathbb{C}[x]$: $F_0'M=\langle x\rangle$, $F_1'M=\langle 1,x,x^2\rangle$, $F_2'M=\langle 1,x,x^2,x^3\rangle$, ... Then $\mathrm{gr}^{F'}M$ looks like this (denoting image of x^i in $\mathrm{gr}^{F'}M$ by v_i):

0	1	2	3
v_1	v_0	v_3	v_4
	v_2		

and ~~$\xi v_i\neq 0$, $xv_0\neq 0$, $xv_1=v_2$, $xv_2=v_3$ ...~~

$\xi v_i=0$, $i\neq 1$, $\xi v_1 = v_0$

$xv_i=v_{i+1}$, $i\neq 0$, $xv_0=0$

So $I_{F'}=(\xi^2,\xi x)$.

<u>Proposition 8.</u> If $Z\subset \mathrm{Supp}\,\mathrm{gr}^F M$ is a component, $\dim Z=d(M)$, and c_Z is its multiplicity, then c_Z is independent of the filtration.

<u>Proof.</u> The proof follows from Corollary 2 in the Appendix on the Jantzen filtration. Indeed, Cor. 2. of Appendix implies that for two good filtrations F, F' on M, $\mathrm{gr}^F M$ and $\mathrm{gr}^{F'} M$ have finite filtrations with the same quotients occurring in the opposite orders. So the ranks of these modules over each components are the same.

<u>Remark.</u> In fact L. 6 and Prop. 7 also follow from this argument.

<u>Ex.</u> In the previous example, the filtration F' we defined satisfies $F_i M \subseteq F'_i M \subseteq F_{i+1} M$, so we are supposed to have Jantzen filtration of length 2. And indeed, $\mathrm{gr}^F M = \mathbb{C}[x]$ has filtrations with quotients $\mathbb{C}[x]$, $\mathbb{C}$, while $\mathrm{gr}^{F'} M$ has the form

sub.

$$v_0 \xleftrightarrow{y} v_1 \xrightarrow{x} v_2 \xrightarrow{x} v_3 \xrightarrow{x} v_4 \to \cdots$$

so has filtration with quotients $\mathbb{C}$, $\mathbb{C}[x]$.

Consider now the geometric filtration on $\mathcal{D}$ and let M be any finitely generated $\mathcal{D}$-module.

<u>Definition.</u> The support of $\mathrm{gr}\, M$ with respect to a good filtration compatible with the geometric filtration of $\mathcal{D}$ is called the singular support of M.

It's denoted $SS(M)$.

The support $\mathrm{supp}(gr^F M)$ for F compatible to the Bernstein filtration is called the arithmetic singular support and denoted $SS_a(M)$.

We'll see later that similarly to $SS_a(M)$, $SS(M)$ is defined as a cycle, i.e. can attach a multiplicity to each irreducible component.

We saw that $SS_a(M)$ is a conical variety. Similarly, $SS(M)$ is conical in the following sense: it's invariant under action of G_m given by $(x,\xi) \to (x, \lambda\xi)$, $\lambda \in k^\times$.

<u>Theorem</u>. $\dim SS(M) = \dim SS_a(M)$ (i.e. $= d(M)$).

We will prove it later.

<u>Ex</u> Let $M = \langle e^{x^2/2} \rangle$. Then $M = \mathcal{D}/\mathcal{D}(\partial - x)$.

So $gr M$ for Bernstein filtration is $\mathbb{C}[x,\xi]/(\xi - x)$, and $SS_a(M)$ is the line $\xi = x$.

But $gr M$ for geometric filtration is $\mathbb{C}[x,\xi]/(\xi)$, so $SS(M) = \{\xi = 0\}$. I.e. they are different.

<u>Remark</u>. So if either dimension $= n$, then M is holonomic.

Appendix to Lecture 3.

Jantzen filtration.

We will need a piece of linear algebra called Jantzen filtration. To define it, assume that we have a morphism of $k[[t]]$-modules $f: V[[t]] \to W[[t]]$, where V, W are k-vector spaces. Suppose that f is injective and Coker f is annihilated by t^s. Then we get a filtration $\{V_i\}$ of V, $V = V_0 \supset V_1 \supset \dots \supset V_s \supset V_{s+1} = 0$ and a filtration $\{W_i\}$ of W, $0 = W_{-1} \subset W_0 \subset \dots \subset W_s = W$, which are called the Jantzen filtrations. They are defined as follows. We set $V_1 = \operatorname{Ker} f_0$ and $W_1 = \operatorname{Im} f_0$. Note that we have an isomorphism induced by f_0: $\varphi_0 : V_0/V_1 \xrightarrow{\sim} W_0$. Now we have a map $\varphi_1 : V_1 \to W/W_0$ defined as follows. Pick $v \in V_1$ and choose a lift $\tilde{v}$ of v to V. Then $f(\tilde{v}) \in tW[[t]]$, so we set $\varphi_1(v)$ to be the image of $\frac{1}{t} f(\tilde{v})$ in W/W_0. This is well defined: if $\hat{\tilde{v}}$ is another lift, then $\hat{\tilde{v}} = \tilde{v} + tu$, so $\frac{1}{t} f(\hat{\tilde{v}}) = \frac{1}{t} f(\tilde{v}) + f(u)$, and the images of $\frac{1}{t} f(\tilde{v})$ and $\frac{1}{t} f(\hat{\tilde{v}})$ in W/W_0 are the same, as $f(u) \bmod t$ is in W_0. Define $W_1 \subset W$ to be the preimage in W of $\operatorname{Im} \varphi_1 \subset W/W_0$. Define $V_2 \subset V_1$ to be $\operatorname{Ker} \varphi_1$.

Note that we have an isomorphism induced by f_1:

$\varphi_1 : V_1/V_2 \xrightarrow{\sim} W_1/W_0$. Now, we have a map $f_2 : V_2 \to W/W_1$ defined as follows.
Pick $v \in V_2$ and choose a lift $\tilde{v}$ of v to V such that $f(\tilde{v}) \in t^2 W[[t]]$. (Such exists: indeed, if $\tilde{\tilde{v}}$ is any lift, then $\underbrace{f(\tilde{\tilde{v}})}_{=tw_0} \in tW_0 \mod t^2 W[[t]]$. Pick $v_0 \in V_0$ such that $f_0(v_0) = w_0$, and set $\tilde{v} = \tilde{\tilde{v}} - tv_0$. Then $f(\tilde{v}) \in t^2 W[[t]]$). Now set $f_2(v)$ to be the image of $\frac{1}{t^2} f(\tilde{v})$ in W/W_1. This is well defined: if $\tilde{\tilde{v}}$ is another lift as above, then $\tilde{v} = \tilde{\tilde{v}} + tu$, $u \in V_1$, so $\frac{1}{t^2} f(\tilde{v}) = \frac{1}{t^2} f(\tilde{\tilde{v}}) + \frac{1}{t} f(u)$, and $\frac{1}{t} f(u) \in W_1$. Define $W_2 \subset W$ to be the preimage in W of $\operatorname{Im} f_2 \subset W/W_1$. Define $V_3 \subset V_2$ to be $\operatorname{Ker} f_2$. Note that we have an isomorphism induced by f_2: $\varphi_2 : V_2/V_3 \xrightarrow{\sim} W_2/W_1$. And we continue like this for s steps, at which point everything ends, as t^s kills coker f.

As a result we obtain the filtrations as stated above, and also isomorphisms between successive quotients:

$\varphi_i : V_i/V_{i+1} \xrightarrow{\sim} W_i/W_{i-1}$. (Note that $V_i = \{ v \in V_0,\ v$ has a lift $\tilde{v}$ with $f(\tilde{v}) \vdots t^i \}$, and W_i is defined in the same way using duality.)

This can be applied to representation theory as follows.

Suppose that A is an algebra which is a flat deformation over $k[[t]]$ of $\overline{A}_0 = A/tA$ (i.e. $A \cong \overline{A}_0[[t]]$ as a $k[[t]]$-module) and V, W are A-modules which are flat deformations of $\overline{V}_0 = V/tV$ and $\overline{W} = W/tW$. Suppose that $f: V \to W$ is a morphism, which becomes an isomorphism after inverting t, $t^s \mathrm{Coker} f = 0$

<u>Corollary 1.</u> Then $\overline{V}, \overline{W}$ admit finite filtrations (of length $\le s+1$), whose successive quotients are the same, but occur in opposite orders.

<u>Proof</u>: As above, noting that all the spaces we considered are now $\overline{A}$-modules.

<u>Corollary 2</u>. Suppose that A is a filtered algebra, and M is an A-module with two filtrations F, F' such that $F'_{i-s}M \subset F_i M \subset F'_{i+s}M$. Then the $\mathrm{gr} A$-modules $\mathrm{gr}^F M$ and $\mathrm{gr}^{F'} M$ admit finite filtrations with the same quotients occurring in the opposite order.

<u>Pf.</u> Let $\hat{A}$ be the completed Rees algebra of A ($\hat{A} = \{a_0 + ta_1 + t^2 a_2 + \dots \mid a_i \in F_i A\}$). Then $\hat{A}/t\hat{A} = \mathrm{gr} A$. Let $\hat{M}, \hat{M}'$ be the completed Rees modules of M, constructed similarly

Obviously, if $\alpha \in \hat{M}$ then $t^s\alpha \in \hat{M}'$, and vice versa, so we have morphisms

$$\begin{tikzcd} \hat{M} \arrow[r, bend left, "f"] & \hat{M}' \arrow[l, bend left, "f'"] \end{tikzcd}$$

such that $f' \circ f = t^{2s}$ $\left(\begin{array}{l} f(\alpha) = t^s\alpha \\ f'(\alpha) = t^s\alpha \end{array} \right)$.

So we can apply Corollary 1, and we get the statement.