Lecture 20.

D-modules with regular singularities (contd)

More on the Riemann-Hilbert map.

In the last lecture we considered the RH map

$$\text{RH}: \begin{array}{c}\text{Algebraic vector bundles}\\ \text{on } X \text{ with a (flat) connection}\\ \text{with RS}\end{array} \longrightarrow \text{Rep}\, \pi_1(X)$$

which attaches to each bundle (E, ∇) the monodromy representation of ∇. Note that both categories, for fixed rank, have a moduli space of objects, which is (generically) an algebraic variety, hence a complex manifold. But the map RH is not algebraic but only holomorphic. Let's consider two examples of what this map does.

Ex.1. X = projective curve. Then the moduli space $\mathcal{M}_{DR}$ of line bundles with connection looks as follows:

We have a map $\mathcal{M}_{DR} \longrightarrow \text{Pic}_0(X) = \text{Jac}(X)$ (Jacobian) whose fiber is $\mathbb{A}^g$ (an affine space bundle). This $\mathbb{A}^g$, where $g = \text{genus}(X)$, is a torsor over $H^0(X, \Omega)$. So it's an algebraic variety of dimension $2g$.

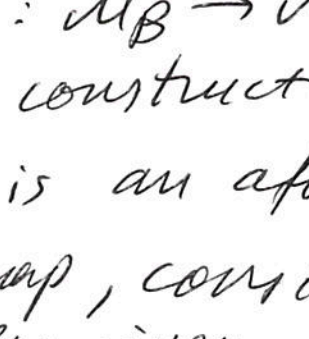
On the other hand, $\mathcal{M}_B$ (Betti moduli space), the moduli space of representations of $\pi_1(X)$, is $(\mathbb{C}^*)^{2g}$, once we fix generators $a_1, \dots, a_g, b_1, \dots, b_g$ of $\pi_1(X)$, $\prod [a_i, b_i] = 1$.

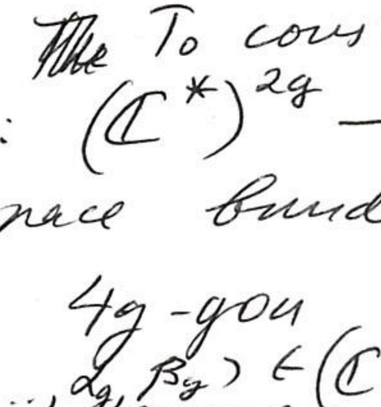
The RH map is a holomorphic isomorph. $\mathcal{M}_{DR} \to \mathcal{M}_B$. Clearly it cannot be algebraic, since ~~any~~ we have

<u>Lemma.</u> Any ~~holomorphic function~~ regular map $\mathbb{C}^* \to J(X)$ is constant.

<u>Pf.</u> This map must extend to $\mathbb{CP}^1 \to J(X)$ and then going to univ. cover $\mathbb{CP}^1 \to \mathbb{C}^n$, which is constant by Liouville thm.

This map looks as follows: RH is inverse to $f: \mathcal{M}_B \to \mathcal{M}_{DR}$. ~~The~~ To construct f, let's construct $\pi: (\mathbb{C}^*)^{2g} \to J(X) = \mathrm{Pic}_0(X)$, which is an affine space bundle. To define this map, consider 4g-gon and ~~glue~~ given $(\alpha_1, \beta_1, \dots, \alpha_g, \beta_g) \in (\mathbb{C}^*)^{2g}$,





glue a line bundle by using α_1 along a_1, β_1 along b_1, etc. (out of the trivial bundle on the polygon).

<u>Ex 2.</u> Let $X = \mathbb{P}^1 \setminus \{0, 1, \lambda, \infty\}$, $\lambda \neq 0, 1, \infty$:

~~Then $\mathcal{M}_{DR}$~~

Let's restrict to connections with trivial determinant. ~~Then~~ Then $\mathcal{M}_{DR}$ has an open set $\mathring{\mathcal{M}}_{DR}$ of connections which have first order poles on the trivial bundle. Also let's fix local monodromy. Namely, $\mathring{\mathcal{M}}_{DR}$ is the set of connections

$$D = \partial - \frac{a_0}{z} - \frac{a_1}{z-1} - \frac{a_\lambda}{z-\lambda}, \quad Tr(a_j) = 0.$$

Let $a_\infty = -a_0 - a_1 - a_\lambda$. Let

$$\mathring{\mathcal{M}}_{DR}(\alpha_0, \alpha_1, \alpha_\lambda, \alpha_\infty) = \left\{ D / \ a_j \sim \begin{pmatrix} \alpha_j & 0 \\ 0 & -\alpha_j \end{pmatrix} \right\}, \ \alpha_j \neq$$

Then $RH: \mathring{\mathcal{M}}_{DR}(\alpha_0, \alpha_1, \alpha_\lambda, \alpha_\infty) \longrightarrow \mathcal{M}_B(\alpha_0, \alpha_1, \alpha_\lambda$

where $\mathcal{M}_B(\alpha_0, \alpha_1, \alpha_\lambda, \alpha_\infty) = \{ A_0, A_1, A_\lambda, A_\infty,$

$$A_0 A_1 A_\lambda A_\infty = 1$$

$$A_j \sim \begin{pmatrix} \exp(2\pi i \alpha_j) & 0 \\ 0 & \exp(-2\pi i \alpha_j) \end{pmatrix} \}$$

This map is highly transcendental.

Namely, let $P \in \mathcal{M}_B(\alpha_0, \alpha_1, \alpha_\lambda, \alpha_\infty)$ and consider the point $Q_\lambda = RH_\lambda^{-1}(P) \in \mathring{\mathcal{M}}_{DR}(\alpha_0, \alpha_1, \alpha_\lambda, \alpha_\infty)$. This defines a flow on $\mathring{\mathcal{M}}_{DR}(\alpha_0, \alpha_1, \alpha_\lambda, \alpha_\infty)$ which is called the Painlevé-6 flow.

Holonomic D-modules with RS in higher dimensions

Constructible sheaves and complexes

Let X be a $\mathbb{C}$-algebraic variety. Denote by X^{an} the corr. analytic variety, cons. in classical topology.

Let $\mathbb{C}_X$ be the const. sheaf on X^{an}. Let $Sh(X^{an})$ be the category of $\mathbb{C}_X$-modules i.e. sheaves of $\mathbb{C}$-vector spaces. The derived cat. of bdd cplxs will be denoted $\mathcal{D}(X^{an})$ ($\mathbb{C}_X$-complexes).

<u>Def</u>. A $\mathbb{C}_X$-module F is constructible if $\exists$ a stratif $X = \cup X_i$ of X by loc. closed alg. subvarieties s.t. $F|_{X_i^{an}}$ is a locally constant complex of f.d. vector spaces.

Also $\mathbb{C}_X$-complex is constr. if all its cohomology sheaves are constr. as $\mathbb{C}_X$-mod. The full subcategory of $D(X_{an})$ consisting of constr. complexes will be denoted by $D_{con}(X_{an})$.

Any morphism $\pi : X \to Y$ of alg. varieties induces a cont. map $\pi^{an} : X^{an} \to Y^{an}$ and we can consider functors

$$\pi_!, \pi_* : D(X^{an}) \to D(Y^{an})$$

$$\pi^*, \pi^! : D(Y^{an}) \to D(X^{an})$$

Also we have $\mathbb{D} : D(X^{an}) \longrightarrow D(X^{an})$.

<u>Theorem.</u> These functors preserve subcat D_{con}, and on them we have ~~formulas~~ $\mathbb{D}^2 = Id$, $\mathbb{D}\pi^*\mathbb{D} = \pi^!$, $\mathbb{D}\pi_*\mathbb{D} = \pi_!$ $\quad (\mathbb{D}M = \underline{RHom}(M, \mathbb{C}_X))$.

<u>De Rham functor.</u> Let $\mathcal{O}_X^{an}$ be the structure sheaf of X^{an}. We will assign to each $\mathcal{O}_X$-module M the corr. "analytic" sheaf of $\mathcal{O}_X^{an}$-modules M^{an}, which is locally given by $M^{an} = \mathcal{O}_{X^{an}} \underset{\mathcal{O}_X}{\otimes} M$.

This defines an exact functor

$an : \mathcal{M}(\mathcal{O}_X) \to \mathcal{M}(\mathcal{O}_{X^{an}})$, and

in particular an exact functor

$an : \mathcal{M}(D_X) \to \mathcal{M}(\mathcal{D}_X^{an})$

($\mathcal{D}_X^{an}$ is the sheaf of analytic diff. operators).

<u>Def.</u> The De Rham functor

$DR : \mathcal{D}^b(D_X) \to D^b(X^{an}) = D^b(Sh(X^{an}))$

is $$DR(M^\cdot) = \Omega_X^{an} \otimes_{\mathcal{D}_X^{an}} (M^\cdot)^{an}$$

<u>Rem.</u> Since $dR(\mathcal{D}_X)$ is a locally proj. resolution of Ω_X, we have

$$DR(M^\cdot) = dR(D_X^{an}) \otimes_{\mathcal{D}_X^{an}} (M^\cdot)^{an}[n]$$

$n = \dim X$.

In partic., if M is an $\mathcal{O}$-coherent $\mathcal{D}_X$-mod. corr. to some vector bdle with a flat connection and $\mathcal{L} = M^{flat}$ is the local system of flat sections of F

Then $DR(M) = \mathcal{L}[n]$

(by Poincare Lemma).

Here is the main theorem about connection between $\mathcal{D}$-modules and constructible sheaves.

Thm. (a) $DR(\mathcal{D}_{hol}(D_X)) \subset \mathcal{D}_{con}(X^{an})$.
Also on $\mathcal{D}_{hol}$ DR commutes with $\mathbb{D}$
Also $\mathbb{D}R$ commutes with tensor product.
(b) On the subcategory D_{rs} the functor $\mathbb{D}R$ commutes with all functors
(c) $DR: \ D_{rs}(D_X) \to D_{con}(X_{an})$
is an equivalence.

Def. Let M be a vector bdle on X with a flat conn. M ~~$\in \mathcal{D}_{hol}(D_X)$~~ lies in $\mathcal{D}_{rs}$ if ~~all~~ restriction of M to any curve has RS.

Def. An irreducible M in $\mathcal{D}_{hol}$ has RS if $M = j_{!*}L$, L a vector bundle with flat conn. and RS.
An object $M \in \mathcal{D}_{hol}$ has RS $\Leftrightarrow$ all comp. factors have RS.