Lecture 2.

Let A be a filtered algebra such that $\operatorname{gr} A \cong \mathbb{C}[y_1, \dots, y_m]$. ($\deg y_i = 1$)

Let M be an A-module with a good filtration F. Let $h_F(M, j) = \dim F_j M$.

<u>Thm 1</u> There exists a polynomial $h_F(M)(t)$, the Hilbert polynomial of M under F, such that $h_F(M, j) = h_F(M)(j)$ for $j \gg 0$. It has the form $h_F(M)(t) = \frac{c t^d}{d!} + \text{lower terms}$, where $d \le m$ and $c \in \mathbb{Z}_+$.

<u>Pf.</u> This follows from the Hilbert syzygy theorem in commutative algebra. (and normalization lemma)

<u>Lemma 2.</u> c and d don't depend on the filtration

<u>Pf.</u> As we proved in the last lecture, ~~the~~ any two good filtrations are equivalent, which implies that for some j_0, j_1

$$h_F(M)(j + j_1) \ge h_{F'}(M)(j) \ge h_F(M)(j - j_0),$$

i.e. c and d are the same.

<u>Def.</u> $d = d(M)$ is called the dimension of M (sometimes Gelfand–Kirillov, or functional dimension).

Theorem 3. (Bernstein inequality)
For every finitely generated module $M \neq 0$ over $\mathcal{D} = \mathcal{D}(\mathbb{A}_n)$ with Bernstein filtration, we have $d(M) \geq n$.

Before proving this theorem, let us derive some important corollaries of this theorem, and show how it implies the results about p^λ.

Ex. 1. Suppose $n = 1$. Then to prove theorem 3 we need to show that $\dim M = \infty$. But this is clear since $[\partial, x] = 1$ (by taking the trace).

2. $M = \mathcal{O} = \mathbb{C}[x_1, \dots, x_n]$ (it is irreducible). Then for the obvious filtration
$$h_F(M)(t) = \binom{n+t}{n} = \frac{(t+n)(t+n-1)\dots(t+1)}{n!}.$$
So $d(M) = n$, $c = 1$.

3. $a \in \mathbb{C}$, δ_a – $\mathcal{D}$-module with basis $\delta_a^{(0)}, \delta_a^{(1)}, \dots$ with action of $\mathcal{D}$ as follows: (also irreducible)
$$\frac{d}{dx} \delta_a^{(k)} = \delta_a^{(k+1)}$$
$$(x-a)\delta_a^{(k)} = k\delta_a^{(k-1)}.$$
$$(x-a)\delta_a^{(0)} = 0$$

This is the $\mathcal{D}$-module formed by the δ-function at a and its derivatives. It's easy to check that $d(\delta_a)=1$, $c(\delta_a)=1$.

<u>Def.</u> If $d(M)=n$ or $M=0$ then M is called holonomic.

<u>Ex.</u> $\mathcal{O}$ and δ_a are holonomic.

<u>Remark.</u> For $n \geq 2$ there exist irreducible non-holonomic $\mathcal{D}$-modules (of $d(M)=2n-1$).

Let $0 \to M_1 \to M_2 \to M_3 \to 0$ be a SES of modules over an algebra A with Noetherian $\operatorname{gr} A$. Then any good filtration on M_2 gives rise to a good filtration on M_1 and M_3.

<u>Proposition 4.</u> 1) $d(M_2)=\max(d(M_1), d(M_3))$

2) If $d(M_2)=d(M_1)=d(M_3)$ then $c(M_2)=c(M_1)+c(M_3)$

3) If $d(M_1)>d(M_3)$ then $c(M_2)=c(M_1)$ and if $d(M_3)>d(M_1)$ then $c(M_2)=c(M_3)$.

<u>Pf.</u> We have $h_F(M_2,j)=h_F(M_1,j)+h_F(M_3,j)$ which implies all the statements.

<u>Cor. 5</u> Let M be a holonomic $\mathcal{D}$-module and $c = c(M)$. Then the length of M is $\leq c$.

<u>Proof.</u> Let M be holonomic, so $d(M) = n$. Let
$$0 \to N \to M \to N' \to 0$$
be an exact sequence. Then by Bernstein inequality $d(N) = d(N') = n$, so $= c(M) = c(N) + c(N')$. So the statement follows by induction in c.

<u>Ex.</u> Length(M) can be smaller than $c(M)$. E.g. Consider $\lambda \notin \mathbb{Z}$, and for $n=1$ $M = \langle x^{\lambda+m}, m \in \mathbb{Z} \rangle$. Then $d(M) = 1$, $c(M) = 2$ (exercise), but M is irreducible. (If $\lambda \in \mathbb{Z}$, then M is reducible and has length 2).

<u>Cor. 6.</u> Let M be a $\mathcal{D}$-module on $\mathbb{A}^n$ with a filtration, and $h \in \mathbb{R}[t]$,
$$h(t) = \frac{ct^n}{n!} + \text{l.o.t.}, \quad c > 0,$$
and
$$\dim F_j M \leq h(j), \quad j \gg 0.$$
Then M is holonomic and length$(M) \leq c$.

<u>Pf.</u> Let N be a f.g. submodule of M. Let's prove that N is holonomic and ...

let F' be a good filtration of N such that $F_j'N \subseteq F_jN$ (such exists, e.g. choose j_0 s.t. $F_{j_0}N$ generates N, and set $F_i'N = F_iN$ for $i \le j_0$, and $F_i'N = F_{i-j_0}\mathcal{D} \cdot F_{j_0}N$ for $i > j_0$). Then $\dim F_j'N \le \dim F_jN$, so $h_{F'}(N)(j) \le \frac{cj^n}{n!} + \text{l.o.t.}$

But by Bernstein inequality, $d(N) \ge n$, hence $h_{F'}(N)(j) = \frac{c'j^n}{n!} + \text{l.o.t.}$ where $c' \le c$, so N is holonomic and has length $\le c'$.

Using the same argument as above, we can show that M has finite length, hence f.g. and holonomic with length $\le c$.

All this theory works if $\mathbb{C}$ is replaced with any field of characteristic zero.

Let $p \in \mathbb{C}[x_1, \dots, x_n]$ and recall that in the last lecture we defined the $\mathcal{D}$-module $M(p) = \{qp^\lambda\}$.

<u>Theorem 7.</u> $M(p)$ is holonomic. In particular, it is finitely generated.

We showed that this implies desired results on p^λ.

<u>Proof of Theorem 7.</u> By Cor. 6 it's enough to find a filtration on $M(p)$ with $\dim F_j M \le h(j)$, where h is a polynomial of degree n.

Let $F_j M(p) = \{ q p^{\lambda - j},\ \deg q \le j(m+1) \}$, $m = \deg p$.

Let us show that this is a filtration. Clearly, $F_{j-1} \subset F_j$, $\bigcup F_j = M$. It's enough to show that $D_1 F_j \subseteq F_{j+1}$.

1) $\forall i$ $\quad x_i q p^{\lambda-j} = x_i q p \cdot p^{\lambda-j-1}$, and $\deg(x_i q p) \le j(m+1) + m+1 = (j+1)(m+1)$

2) $\partial_i (q p^{\lambda - j}) = (\partial_i q) \cdot p^{\lambda - j} + (\lambda - j) q \partial_i p \cdot p^{\lambda - j - 1}$,
$\deg(q \partial_i p) \le j(m+1) + m - 1 \le (j+1)(m+1)$. So F is a filtration.

We have $\dim F_j M = \binom{j(m+1)+n}{n}$, so it's a polynomial of degree n in j.

By Cor 6, M is holonomic.

It remains to prove the Bernstein inequality. We begin with the following lemma.

<u>Lemma 8.</u> Let M be a $\mathcal{D}$-module with a good filtration, $F_0 M \ne 0$.

Then the natural map

$$F_i\mathcal{D} \longrightarrow Hom(F_iM, F_{2i}M)$$

is an embedding for any i.

<u>Proof</u>. We will prove the statement by induction in i. For $i=0$ it is clear. Suppose the statement is true for all $i'<i$. Let $a \in F_i\mathcal{D}$ be such that

$$a = \sum_{i_1 \le \dots \le i_k} p_{i_1\dots i_k} \partial_{i_1} \dots \partial_{i_k}.$$

We may assume that a is not a scalar. Suppose ∂_m occurs with a nonzero coefficient. Then $[a, x_m] \neq 0$. Similarly, if x_m occurs with a nonzero coefficient, then $[a, \partial_m] \neq 0$.

By property of Bernstein's filtration $[a, x_m]$ and $[a, \partial_m]$ are in $F_{i-1}\mathcal{D}$.

Suppose e.g. that $[a, x_m] \neq 0$. (The other case is treated similarly). We need to show that $\exists \alpha \in F_iM$ s.t. $a\alpha \neq 0$. By the induction hypothesis there is $\alpha' \in F_{i-1}M$ s.t. $[a, x_m]\alpha' \neq 0$. So $a(x_m\alpha') - x_m(a\alpha') \neq 0$.

Thus either $a(x_m \alpha') \neq 0$ or $a\alpha' \neq 0$, and we can take $\alpha = x_m \alpha'$ or $\alpha = \alpha'$. ∎

Now we deduce Bernstein's inequality from the lemma. We know that

$$\dim F_i \mathcal{D} = \frac{i^{2n}}{(2n)!} + \text{lower terms}.$$

But by the lemma, $\dim F_i \mathcal{D} \leq \dim \operatorname{Hom}(F_i M_F, F_{2i} M)$

$= h_F(M, i) h_F(M, 2i)$. So

$$\frac{i^{2n}}{(2n)!} + \text{l.o.t.} \leq C^2 \frac{i^d (2i)^d}{d!^2} + \text{l.o.t.}, \text{ hence}$$

$n \leq d$. ∎