lecture 2.

Let \( A \) be a filtered algebra such that \( \text{gr} A = \mathbb{C}[y_1, \ldots, y_m] \) (\( \deg y_i = 1 \)). Let \( M \) be an \( A \)-module with a good filtration \( F \). Let \( h_F(M,j) = \dim F_j M \).

**Thm.** There exists a polynomial \( h_F(M)(t) \). The Hilbert polynomial of \( M \) under \( F \) such that \( h_F(M,j) = h_F(M)(j) \) for \( j \gg 0 \).

It has the form \( h_F(M)(t) = \frac{ct^d}{d!} + \text{lower terms} \)

where \( d \leq m \) and \( c \in \mathbb{Z}_+ \).

**Pf.** This follows from the Hilbert syzygy theorem in commutative algebra (and normalization lemma). (Lemma 2. \( c \) and \( d \) don't depend on the filtration.

**Pf.** As we proved in the last lecture, for any two good filtrations are equivalent, which implies that for some \( j_0, j_1 \)

\[ h_F(M)(j+j_1) \geq h_F(M)(j) \geq h_F(M)(j-j_0), \]

i.e. \( c \) and \( d \) are the same.

**Def.** \( d = d(M) \) is called the dimension of \( M \) (sometimes Aldand-Kirillov or functional dimension).
Theorem 3. (Bernstein inequality)

For every finitely generated module $M^0 \in \mathcal{D} = \mathcal{D}(A_n)$ with Bernstein filtration, we have $d(M) \geq n$.

Before proving this theorem, let us derive some important corollaries of this theorem, and show how it implies the results about $p^1$.

**Ex. 1.** Suppose $n = 1$. Then to prove theorem 3 we need to show that $\dim M = \infty$. But this is clear since $[D, x] = 1$ (by taking the trace).

2. $M = \mathbb{C}[x_1, \ldots, x_n]$. Then for the obvious filtration

$$h_f(M)(t) = \binom{n + t}{n} = \frac{(t+n)(t+n-1)\cdots(t+1)}{n!}$$

So $d(M) = n$, $c = 1$.

3. $a \in \mathbb{C}$, $\delta^a - \mathcal{D}$-module with basis $\delta^{(0)}_a, \delta^{(1)}_a$, \ldots with action of $\mathcal{D}$ as follows:

$$\frac{d}{dx} \delta^{(k)}_a = \delta^{(k+1)}_a$$

$$\quad (x-a) \delta^{(k)}_a = \delta^{(k)}_a - \delta^{(k-1)}_a$$

$$\quad (x-a)^n \delta^{(0)}_a = 0$$

(also irreducible)
This is the D-module formed by the δ-function at a and its derivatives. It's easy to check that $d(\delta_a) = 1$, $c(\delta_a) = 1$.

Def. If $d(M) = n$, then $M$ is called holonomic.

Ex. $\theta$ and $\delta_a$ are holonomic.

Remark. For $n \geq 2$, there exist irreducible non-holonomic D-modules (of $d(M) = 2n - 1$).

let $0 \rightarrow M_1 \rightarrow M_2 \rightarrow M_3 \rightarrow 0$ be a SES of modules over an algebra $A$ with Noetherian grade. Then any good filtration on $M_2$ gives rise to a good filtration on $M_1$ and $M_3$.

Proposition 4. 1) $d(M_2) = \max\{d(M_1), d(M_3)\}$

2) If $d(M_2) = d(M_1) = d(M_3)$ then $c(M_2) = c(M_1) + c(M_3)$

3) If $d(M_1) > d(M_3)$ then $c(M_2) = c(M_1)$ and if $d(M_3) > d(M_1)$ then $c(M_2) = c(M_3)$

pf. We have $h_F(M_2, j) = h_F(M_1, j) + h_F(M_3, j)$ which implies all the statements.
Cor. 5 Let $M$ be a holonomic $D$-module and $c = c(M)$. Then the length of $M$ is $\leq c$.

Proof. Let $M$ be holonomic, so $d(M) = n$. Let $0 \to N \to M \to N' \to 0$ be an exact sequence. Then by Bernstein inequality $d(N) = d(N') = n$, so $c(M) = c(N) + c(N')$. So the statement follows by induction in $c$.

Ex. Length $(M)$ can be smaller than $c(M)$. E.g. Consider $\lambda \notin \mathbb{Z}$, and for $n = 1$ $M = \langle x^{\lambda + m}, m \in \mathbb{Z} \rangle$. Then $d(M) = 1$, $c(M) = 2$ (exercise), but $M$ is irreducible. (If $\lambda \in \mathbb{Z}$, then $M$ is reducible and has length 2.)

Cor. 6 Let $M$ be a $D$-module on $\mathbb{A}^n$ with a filtration, and $h \in \mathbb{R}[t]$, $h(t) = \frac{c t^n}{n!} + \text{e.o.f.}$, $c > 0$, and

$$\dim \mathbb{F} \cdot M \leq h(j), \ j \gg 0.$$ 

Then $M$ is holonomic and $\text{length}(M) \leq c$.

Pf. Let $N$ be a f.g. submodule of $M$. It's prove that $N$ is holonomic.
let $F'$ be a good filtration of $N$ such that $F_i' N \subseteq F_i N$ (such exists, e.g. choose $i_0 \in I$ so $F_{i_0} N$ generates $N$, and set $F_i' N = F_i N$ for $i \leq i_0$, and $F_i' N = F_{i_0} \cdot F_i N$ for $i > i_0$). Then $\dim F_j' N \leq \dim F_j N$, so $h_{F_i', (N)}(j) \leq \frac{c_i 1^n}{n!} + o_1$. But by Bernstein inequality, $d(N) \leq n$, hence $h_{F_i', (N)}(j) = \frac{c' 1^n}{n!} + o_1$, where $c' \leq c$. So $N$ is holonomic and has length $\leq c!$

Using the same argument as above, we can show that $M$ has finite length, hence f.g. and holonomic with length $\leq c$.

All this theory works if in $F'$ is replaced with any $\mathbb{F}$-characteristic $\mathcal{F}$.

Let $\mathbb{F} = \mathbb{C}[x_1, \ldots, x_n]$ and recall that in the last lecture we defined the $\mathcal{D}$-module $M(\mathbb{F}) = \{ qP^{\lambda} \}$.

**Theorem 7.** $M(\mathbb{F})$ is holonomic. In particular, it is finitely generated.

We showed that this implies desired results on $\mathfrak{p}^\lambda$. 

Proof of Theorem 7. By Cor. 6 it's enough to find a filtration on $M(p)$ with $\dim F_i M \leq h(j)$, where $h$ is a polynomial of degree $n$. Let $F_i M(p) = \{ q p^{1-j}, \deg q \leq j(m+1) \}$ let us show that this is a filtration. Clearly, $F_{j-1} \subset F_j$, $UF_j = M$. It's enough to show that $D_1 F_j \subset F_{j+1}$.

1) \( x \cdot q p^{1-j} = x \cdot q p \cdot p^{2-j-1}, \) and
\[ \deg (x \cdot q p) \leq j(m+1) + m + 1 = (j+1)(m+1) \]

2) \( \partial_i (q p^{1-j}) = (\partial_i q) \cdot p^{1-j} + 2 q p^{1-j-1}, \)
\[ \deg (\partial_i q) \leq j(m+1) + M - 1 \leq (j+1)(m+1). \] So $F_j$ is a filtration of $M = (j(m+1) + 1)$, so it's a polynomial of degree $n$ in $j$.

By Cor 6, $M$ is holonomic.

It remains to prove the Bernstein inequality. We begin with the following lemma.

**Lemma 8.** Let $M$ be a $D$-module with a good filtration, $F_i M \neq 0$. 

Then the natural map 
\[ F_i D \to \text{Hom}(F_i M, F_{2i} M) \]
is an embedding for any \( i \).

**Proof.** We will prove the statement by induction in \( i \). For \( i = 0 \) it is clear.
Suppose the statement is true for all \( i' < i \). Let \( a \in F_1 D \) be such that
\[ a = \sum_{i_1 \leq \ldots \leq i_k} p_{i_1, \ldots, i_k} d_{i_1} \cdots d_{i_k}. \]

We may assume that \( a \) is not a scalar. Suppose \( \exists d_m \) occurs with a nonzero coefficient. Then \( [a, x_m] \neq 0 \).
Similarly, if \( \exists x_m \) occurs with a nonzero coefficient, then \( [a, d_m] \neq 0 \).

By property of Bernstein's filtration, \([a, x_m]\) and \([a, d_m]\) are in \( F_{i-1} D \).

Suppose e.g. that \( [a, x_m] \neq 0 \). (The other case is treated similarly.) We need to show that \( \exists d \in F_1 M \) s.t. \( a d \neq 0 \). By the induction hypothesis there is \( d' \in F_{i-1} M \) s.t.
\([a, x_m] d' \neq 0\). So \( a(x_m d') - x_m (a d') \neq 0 \).
Thus either \( d(x_m, x') \neq 0 \) or \( d' \neq 0 \), and we can take \( d = x_m, d' \) or \( d = x' \).

Now we deduce Bernstein's inequality from the lemma. We know that
\[
\dim F_i \frac{D}{D} = \frac{i^{2n}}{(2n)!} + \text{lower terms}.
\]
But by the lemma, \( \dim F_i \frac{D}{D} \leq \dim \text{Hom}_{F^i \frac{D}{D}} \).

\[
= h_{F_i}(M, i) h_{F_i}(M, 2i).
\]
So
\[
\frac{i^{2n}}{(2n)!} + \text{lower terms} \leq c^2 \frac{i d(2i)}{d!} + \text{lower terms},
\]

hence
\[
n \leq d.
\]