Lecture 19.

D-modules with regular singularities

Let X be a C^∞-manifold. Recall that a local system on X consists of the following data:

1) A f.d. vector space V_x for every point $x \in X$

2) An isom $\alpha_\gamma : V_{x_1} \to V_{x_2}$ for every C^∞-path γ from x_1 to x_2.

This data should depend only on the homotopy class of γ and should be compatible with composition of paths.

If X is connected, we may identify the category of local systems on X with the category of f.d. representations of $\pi_1(X, x)$ for some $x \in X$.

Let now X be a complex manifold. Then we have an equivalence of categories "holomorphic vector bundles on X with a flat connection $\cong$ Representations of $\pi_1(X)$ (= local systems on X).

This is not true in the algebraic setting. For example, let $X = \mathbb{A}^1$ and consider the connection on the trivial ~~at~~ bundle given by

$\nabla f(x) = (f'(x) - f(x))dx.$

This connection has a nowhere vanishing section $f = e^x$, so the corresponding local system is trivial, but this connection is not isomorphic to the trivial one as an algebraic connection (as e^x is not a rational function). ~~The~~ Nevertheless, it turns out that we can single out some subcategory of the category of $\mathcal{O}$-coherent $\mathcal{D}$-modules on a smooth algebraic variety X (called ~~[illegible]~~ $\mathcal{D}$-modules with regular singularities) for which this equivalence is still valid. ~~[illegible]~~ Here we will do it for curves.

<u>Regular connections on a disc</u>

First we develop some analytic theory, and then apply it to the algebraic setting. Let $D = \{x \in \mathbb{C} \,/\, |x| < r\}$ and let D^* be the punctured disc. Let $\mathcal{O}_D$ be the algebra of holomorphic functions on D and let $\mathcal{O}_D[x^{-1}]$ be the algebra of meromorphic functions on D holomorphic on D^*.

Also we'll denote by $\Omega_D[x^{-1}]$ the space of merom. 1-forms on D which is holom on D^*. Let Also $\Omega_{log} \subset \Omega_D[x^{-1}]$ be the space of forms of pole order ≤ 1 at 0.

Def. A meromorphic connection on $\mathcal{D}$ (holomorphic on $\mathcal{D}^*$) is a vector bundle M on $\mathcal{D}$ with a connection $\nabla : M \to M \otimes_{\mathcal{O}_{\mathcal{D}}} \Omega_D[x^{-1}]$

By a morphism of merom. connections we mean a morphism $\alpha : M_1 \to M_2$ of vector bundles on $\mathcal{D}^*$ which is merom. at 0, and compatible with connections. Thus, meromorphic connections form a category (not that there is no forgetful functor from this cat. to the cat. of vector bundles on $\mathcal{D}$).

If we choose a ~~holomorphic~~ meromorphic trivialization of M on $\mathcal{D}$ then the connection is given by a matrix $A(z)$ of meromorph. 1-forms. This matrix A is defined uniquely up to gauge transformations $A \to g A g^{-1} + g \cdot \partial(g^{-1})$ where $g : D^* \to GL(n)$ meromorphic.

<u>Definition</u>. We say that ∇ has regular singularities if there exists a trivializati[on] as above such that all entries of A have a pole of order at most 1. More invariantly, (M,∇) is regular holonomic if $(M,\nabla)\cong(M',\nabla')$, where $\nabla': M' \to M' \otimes_{\mathcal{O}} \Omega_{\log}$. In other words, M' is stable under $x\frac{d}{dx}$.

<u>Ex.</u> 1) Any connection with pole of order ≤ 1 is regular. The converse is true if $\operatorname{rank} M = 1$. Indeed, in this case A is a scalar 1-form, and gauge transf. act by $g\circ A = A + g\cdot\partial g^{-1}$. But $g\partial g^{-1} = -\partial \log g$ has a pole of order ≤ 1 at 0. So the condition of having a pole of order ≤ 1 is invariant under gauge transformations. But for rank 2 we have examples of regular connections with poles of order >1. e.g. let $A(z) = \begin{pmatrix} 0 & \beta(z) \\ 0 & 0 \end{pmatrix}$, β any merom. function. Then we claim that $\partial + A$ has regular singularities. Indeed, let $u = \int \tilde{\beta}\, dz$, where $\tilde{\beta}$ is $\beta - c_{-1}(\beta) z^{-1}$, i.e.

we delete the z^{-1} term.

Then $-u' + \beta$ has poles of order ≤ 1,

So taking $g = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}$, we get

$$g \circ A = g A g^{-1} + g\, \partial(g^{-1})$$

$$= \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & \beta \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & -u \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & u' \\ 0 & 0 \end{pmatrix}$$

$= \begin{pmatrix} 0 & \beta - u' \\ 0 & 0 \end{pmatrix}$, which has poles of order ≤ 1.

Properties of RS connections:

1) if ∇_1, ∇_2 have RS then $\nabla_1 \otimes \nabla_2$ has RS
2) ∇ has RS $\Rightarrow$ ∇^* has RS.
3) ∇_1, ∇_2 has RS $\Rightarrow$ $\underline{Hom}(\nabla_1, \nabla_2)$ has RS

Here is an analytic characterization of connections with RS.

<u>Def.</u> Let f be a vector-valued function defined in some sector $\left\{ z = \rho e^{i\theta} \,\middle|\, \begin{matrix} 0 < |\rho| < r \\ \alpha < \theta < \beta \end{matrix} \right\}$

We say that f has moderate growth (or power growth) near 0 if $\exists\, C, \gamma$ s.t.

$$\| f(\rho e^{i\theta}) \| \leq C \rho^{-\gamma}$$

<u>Theorem.</u> A meromorphic connection ∇ is regular if and only if for every sector $\alpha < \arg(x) < \beta$ the horizontal

sections of D on this sector have moderate growth.

Proof. Suppose D is regular. Then we are looking for asymptotics of solutions of $\frac{dF}{dz} = A(z)F$ where $A(z)$ is a matrix with pole of order ≤ 1. Let $\tilde{A}(z) = zA(z)$ (F: sector $\to \mathbb{C}^n$)

Then $\tilde{A}$ is regular at $z=0$, and we have $\rho \frac{dF}{d\rho} = \tilde{A}(\rho e^{i\theta})F$.

Lemma. Let $f'(t) = B(t)f(t)$ on $[0,L]$. $B \in C'[0,L]$.

Then $\|f(L)\| \le \|f(0)\| e^{L \max \|B\|}$.

Pf. $f(L) = \lim_{n\to\infty} \prod_{j=0}^{n-1} e^{\frac{L}{n}B(\frac{j}{n})} f(0)$

Let $M = \max\|B\|$. So

$$\|f(L)\| \le \limsup_{n\to\infty} \prod_{j=0}^{n-1} \|e^{\frac{L}{n}B(\frac{j}{n})}\| \cdot \|f(0)\| \le$$

$$\le \prod_{j=0}^{n-1} e^{\frac{L}{n}M} \|f(0)\|$$

$$= e^{LM}\|f(0)\|.$$

Now we set $t = \log\rho$ and obtain the required estimate for solutions (So $\gamma = \sup\|B\|$ or $\|B(0)\|+\varepsilon$.)

Now assume solutions of ∇ have moderate growth. Consider the restriction of ∇ to D^*. This is a holomorphic bundle with connection on D^*, so it has some monodromy matrix $A \in GL_n(\mathbb{C})$. Pick $a \in \mathfrak{gl}_n(\mathbb{C})$ such that $A = e^{2\pi i a}$. Then the connection $\nabla_0 : \partial - \frac{a}{z}$ has the same monodromy. So $\exists$ an isomorphism $\varphi : \nabla_0 \to \nabla$. This is a flat section of $\nabla_0^* \otimes \nabla^*$, whose solutions have moderate growth (as they are products of solutions of ∇_0^* and ∇). Also φ is single valued. So φ is meromorphic. Hence ∇ can be brought by gauge transf. to the form with first order pole. $\blacksquare$

<u>Corollary</u>. Restriction to D^* is an equiv. of categories Regular merom. connections $\xrightarrow{F}$ $\to$ holom. connections on D^*.

<u>Pf.</u> Clear, F is exact, faithful. Also F is essentially surjective, as any $A \in GL_n(\mathbb{C})$ is of the form $e^{2\pi i A}$, and so A is defined by $\nabla_0 = \partial - \frac{a}{z}$. So we only need to check

that F is full, i.e. that

$Hom(M_1, M_2) \to Hom(F(M_1), F(M_2))$ is surjective. Let $\varphi: F(M_1), F(M_2)$, then φ is a hol. section of $\underline{Hom}(F(M_1), F(M_2))$ on D^* i.e. a hol section of $\underline{Hom}(M_1, M_2)$ on D^*. It has moderate growth, so meromorphic hence defines a morphism $M_1 \to M_2$.

<u>Regular connections on an arbitrary curve.</u>

Let X be a smooth proj. curve, $j: Y \hookrightarrow X$ open subset, $S = X \setminus Y$. Let $\mathcal{D}_X^S$ be the subsheaf of $\mathcal{D}_X$ generated (locally) by $\mathcal{O}_X$ and vector fields that vanish at S.

<u>Def</u> We say that an $\mathcal{O}_Y$-coherent $\mathcal{D}_Y$-module N has RS if there exists an $\mathcal{O}_X$-coherent (= vector bundle on X) submodule of N which is stable under $\mathcal{D}_X^S$.

~~It is clear that this category is closed under taking subquotients.~~

Let N be a vector bundle on X [illegible] defined away from S.

<u>Prop:</u> $N|_Y$ is an $\mathcal{O}_Y$-coherent RS $\mathcal{D}_Y$-module $\iff$ restriction of N to a disk around every $s \in S$ has regular singularities.

<u>Pf.</u> Assume $N|_{\text{disks around } s\in S}$ has RS, and $g_1, \dots, g_m$ are gauge transformations making poles first order at $s_1, \dots, s_m$. Then we can make an $\mathcal{O}$-module by gluing $N|_Y$ with $\mathcal{O}^r$ around s_i using g_i. This $\mathcal{O}$-module inside $N|_Y$ is stable under $\mathcal{D}_Y^S$. Conversely, if $N|_Y$ has a $\mathcal{O}$-submodule $\bar{N}$, which is coherent and $\mathcal{D}_Y^S$-stable, then trivialize $\bar{N}$ near s_i, this will give a realization of D near s_i which has first order poles. $\square$

<u>Corollary.</u> The category of regular connections on Y is stable under subquotients.

<u>Pf.</u> Follows from the fact that RS is equivalent to moderate growth of solutions.

<u>Thm.</u> The natural functor
$\mathcal{O}$-coherent $\mathcal{D}_Y$-modules with RS $\to$ connections on Y^{an}
is an equivalence of categories.
In particular, we have an equivalence
"$\mathcal{O}$-coherent $\mathcal{D}$-modules with RS" $\to$
$\to$ ~~conn~~ representations of $\pi_1(Y)$.

<u>Pf.</u> Let us denote the functor in question by $N \to N^{an}$ (call it analytification functor). The functor is clearly exact and faithful, so we must show that
1) It's surjective on objects, and
2) $\mathrm{Hom}(N_1, N_2) \to \mathrm{Hom}(N_1^{an}, N_2^{an})$
is surjective.

(2) is proved in the same way as for disk. Let's prove (1).
Let (N^{an}, ∇^{an}) be a holomorphic vector bundle with a connection over Y. We must show that there exists an algebraic vector bundle M on X with a connection ∇ with poles of 1st order at S such that $(M, \nabla)|_Y^{an} \cong (N^{an}, \nabla^{an})$.

By the theorem from before, (M, ∇) exists near each $s \in S$. We can glue these into a holomorphic bundle M on X with a meromorphic connection having poles of 1st order along S.
By GAGA M has a unique algebraic structure and gives rise to an alg. vector bundle M on X. Thus, ∇_{an} is a section of some holomorphic bundle on X and hence is algebraic by GAGA ■

<u>Theorem.</u> The category of RS-modules is stable under extensions. (21-st Hilbert problem)

<u>Pf.</u> It's enough to prove this for the disk. Suppose we have an extension

$$0 \longrightarrow N_1 \longrightarrow N \longrightarrow N_2 \longrightarrow 0.$$

Let's bring N_1 and N_2 to first order form, then we have

$$\frac{dF}{dz} = \frac{\begin{pmatrix} A_1 & B(z) \\ 0 & A_2 \end{pmatrix}}{z} F, \text{ i.e.}$$

$$\frac{dF_1}{dz} = \frac{A_1}{z} F_1$$

$$\frac{dF_2}{dz} = \frac{A_2}{z} F_2 + B(z) F_1$$

So $F_1(z) = z^{A_1}$ (fund. solution),

and $$\frac{dF_2}{dz} = \frac{A_2}{z} F_2 + B(z) z^{A_1}.$$

Let $G = z^{-A_2} F_2$. Then

$$\frac{dG}{dz} = -\frac{A_2}{z} G + \frac{A_2}{z} G + z^{-A_2} B(z) z^{A_1}$$

$$= z^{-A_2} B(z) z^{A_1}$$

So $$G = \int z^{-A_2} B(z) z^{A_1} dz,$$

and all the matrix entries clearly have moderate growth.

<u>Def</u>. A holonomic $\mathcal{D}$-module on a curve X is RS if it is an $\mathcal{O}$-coherent RS module on some dense open set $Y \subset X$.

<u>Ex</u>. Consider holonomic $\mathcal{D}$-modules with RS on $\mathbb{C}$ which are $\mathcal{O}$-coherent on $\mathbb{C}^*$

We can completely describe this category $\mathcal{C}$.

<u>Thm.</u> $\mathcal{C}$ is equivalent to the following category: objects are pairs of f.d. vector spaces E, F with linear maps $u: E \to F$, $v: F \to E$, where the eigenvalues of vu lie in some fundamental domain U of $\mathbb{Z}$ on $\mathbb{C}$ containing 0.

The functor in one direction: If M is a module as above then denote by M^α the generalized α-eigenspace of $x\frac{d}{dx}$. Then we define

$$F = \bigoplus_{\alpha \in U} M^{\alpha - 1}, \quad E = \bigoplus_{\alpha \in U} M^\alpha.$$

We let v to be multiplication by x and $u = \frac{d}{dx}$. The functor in the opposite direction: given (E, F, u, v), define

$$M = \mathbb{C}[x] \otimes E \oplus \mathbb{C}\left[\frac{d}{dx}\right] \otimes F$$

and action of $x, \frac{d}{dx}$ is defined by

$$\frac{d}{dx}(1 \otimes e) = 1 \otimes u(e), \quad x(1 \otimes f) = 1 \otimes v(f)$$

(the action is extended using Leibniz rule and the action of x on $\mathbb{C}[x]\otimes E$ and $\frac{d}{dx}$ on $\mathbb{C}\left[\frac{d}{dx}\right]\otimes F$ are the natural ones). It is easy to check that these two actions are mutually inverse.

<u>Remark</u>. We have $E=\bigoplus_{\beta\in U} E_\beta$, $F=\bigoplus_{\beta\in U} F_\beta$, direct sums of generalized eigenspaces of uv and vu. Clearly, for $\beta\neq 0$,

$E_\beta \underset{v}{\overset{u}{\rightleftarrows}} F_\beta$, so we get

$\mathcal{C}=\mathcal{C}_0\oplus\bigoplus_{\substack{\beta\in U\\ \beta\neq 0}}\mathcal{C}_\beta$, where $\mathcal{C}_\beta\cong\mathbb{C}[[t]]$-modu.

Namely, for a $\mathbb{C}[[t]]$-module E, the corr. $\mathcal{D}$-module $\Phi_\beta(E)$ is the Goresky-Macpherson extension $j_{!*}$ of $\partial-\frac{\beta+t}{z}$. ($t: E\to E$ as nilpotent operator).

$\mathcal{C}_0$ has more interesting structure.

It is the category of repr. of the quiver $\circ \rightleftarrows \circ$ where both compositions are nilpotent. One can show that the indecomposables in this category are extensions

$\ldots\mathcal{O}\delta\mathcal{O}\delta\mathcal{O}\ldots$

which can begin and end with δ and $\mathcal{O}$ (so have 4 types).

i.e.: