Lecture 18.

Theorem. The functors $\pi^!$, π_*, $\mathbb{D}$, $\boxtimes$ preserve the derived category of holonomic $\mathcal{D}$-modules.

Proof. The theorem is clear for $\boxtimes$. For $\mathbb{D}$ this is a local statement, so it reduces to the affine case, and in this case we already know it. For $\pi: X \to Y$, the case of $\pi^!$ reduces to the cases when both X and Y are affine, and in this case we already know it. It remains to show the property for π_*. Any π is a composition of a closed embedding, an open embedding, and a projection $\mathbb{P}^N \times Y \to Y$. The case of closed embedding is local, so reduces to affine case, and we already proved it.

The case of open embedding $j: U \hookrightarrow X$. We may assume that X is affine. Let us reduce to the case when U is affine, in which case we have already proved the statement. Cover U by affine open sets, and

Consider the Čech complex: $U = \bigcup_{\alpha=1}^{n} U_\alpha$

Namely $C_k = \bigoplus_{\{\alpha_1, \dots, \alpha_k\}} j_{\alpha_1, \dots, \alpha_k *} M|_{U_{\alpha_1} \cap \dots \cap U_{\alpha_k}}$.

Then C represents $j_*(M)$ (by definition). We know that C_k are holonomic, so C is holonomic, i.e. $j_* M$ is holonomic.

<u>Case of projection $\pi: \mathbb{P}^N \times Y \to Y$</u>. We may assume that Y is affine, and we already know the statement for $\overline{\pi}: \mathbb{A}^N \times Y \to Y$. So we proceed by induction. Let M on $\mathbb{P}^N \times Y$ be holonomic, and $j: \mathbb{A}^N \times Y \hookrightarrow \mathbb{P}^N \times Y$. Then we have an exact triangle

$M \to j_* M \to N$, where N is supported on $\mathbb{P}^{N-1} \times Y$. So we have

$$\pi_* M \to \pi_* j_* M \to \pi_* N.$$

Now, $\pi_* N$ is holonomic by the induction assumption, and $\pi_* j_* M = (\pi \circ j)_* M$ is holonomic by the affine case. So $\pi_* M$ is holonomic, as desired $\blacksquare$.

<u>Remark</u>. Let $\pi: X \to pt$, where X is smooth, $\dim X = n$. Then $H^i(\pi_* \mathcal{O}) = H^{i+n}(X, \mathbb{C})$. So $H^i(\pi_! \mathcal{O}) = H^{-i}(\pi_* \mathcal{O})^* = H^{n-i}(X, \mathbb{C})^* = H_c^{n+i}(X, \mathbb{C})$ by

Poincaré duality. Therefore $\pi_!$ is called the direct image with compact support, and its right adjoint $\pi^!$ is called the inverse image with compact support.

If H is singular, instead of $\mathcal{O}$ we can take IC_X, then we will get the intersection cohomology IH^*, resp. intersection cohomology with compact support, IH^*_c.
with appropriate shift

Ex. 1. Let $X \subset \mathbb{A}^2$ given by $xy = 0$.

$X = X_1 \cup X_2$, $X_1 = \{x=0\}$, $X_2 = \{y = 0\}$.

We have $IC_X = i_{*1}\mathcal{O} \oplus i_{*2}\mathcal{O}$ $\quad i_1 : X_1 \hookrightarrow X$

(it satisfies the conditions) $\quad i_2 : X_2 \hookrightarrow X$.

So

$$\dim IH^i(X) = \begin{cases} 2, & i = 0 \\ 0 & \text{otherwise}. \end{cases}$$

(so it is different from the ordinary coh.)

2. Let X be smooth and a group Γ, $|\Gamma| < \infty$, acts on X freely at the generic pt.

Let $Y = X/\Gamma$ (assume it's a variety)

Then Γ acts on $\pi_* \mathcal{O}_X$. Let $V \subset Y$ be the open set of points coming from the locus of trivial stabilizer.

Lemma. $j_{!*}\left((\pi_*\mathcal{O}_X)|_V\right) = \pi_*\mathcal{O}_X$.

Proof. Let $\tilde{j} : U \hookrightarrow X$ where $U = \pi^{-1}V$

~~Since Γ acts freely at the general~~ point the map π

Since π is a finite morphism, it is proper. Hence

$$j_!((\pi_* \mathcal{O}_X)|_V) = \pi_*(\tilde{j}_!(\mathcal{O}_X|_U))$$

$$j_*((\pi_* \mathcal{O}_X)|_V) = \pi_*(\tilde{j}_*(\mathcal{O}_X|_U)).$$

So $\quad Im(j_!(\pi_* \mathcal{O}_X|_V) \to j_*(\pi_* \mathcal{O}_X|_V))$

$$= \pi_*(Im(\tilde{j}_! \mathcal{O}_X|_U) \to \tilde{j}_* \mathcal{O}_X|_U)) = \pi_* \mathcal{O}_X.$$

Since X is smooth. And this means that $j_{!*}((\pi_* \mathcal{O}_X)|_V) = \pi_* \mathcal{O}_X$

Corollary. $IC_Y = (\pi_* \mathcal{O}_X)^\Gamma$.

Corollary. $\quad H^i(IC_Y) = IH^i(Y)$

equals $\quad H^{n+i}(Y, \mathbb{C}) = H^{n+i}(X, \mathbb{C})^\Gamma$, where $n = \dim X$.

Ex. $Y \subset \mathbb{C}^3$ defined by $xy - z^2 = 0$.

Then $Y = \mathbb{C}^2 / \mathbb{Z}/2$, so

$IH^*(Y) = H^*(Y)$ (so it coincides with the usual cohomology).

<u>Example</u>. Let $\pi : X \to Y$ be a morphism of irred. algebraic varieties. We say that π is small if $\operatorname{codim}\{y \in Y \mid \dim \pi^{-1}(y) \geq m\} \geq 2m+1$.

<u>Prop.</u> Suppose $\pi : X \to Y$ is a small resolution of singularities. Then $\pi_* \mathcal{O}_X = IC_Y$, so $IH^\bullet(Y) = H^\bullet(X)$.

<u>Example</u>. Let $\mathfrak{g}$ be a simple Lie algebra, and $\tilde{\mathfrak{g}} = \{(x, b) \mid x \in \mathfrak{g},\ b \ni x \text{ a Borel subalgebra}\}$.

We have a map $\pi : \tilde{\mathfrak{g}} \to \mathfrak{g} \times_{\mathfrak{h}/W} \mathfrak{h}$,

$\pi(x, b) = (x, \bar{x})$, where $\bar{x}$ is the image of x in $b/[b,b] = \mathfrak{h} \leftarrow$ Standard Cartan.

It is known that π is a small resolution (called the Grothendieck simultaneous resolution). Thus $IH^*(\mathfrak{g} \times_{\mathfrak{h}/W} \mathfrak{h}) = H^*(\tilde{\mathfrak{g}}) =$
$= H^*(G/B)$.

<u>Ex.</u> $\mathfrak{g} = \mathfrak{sl}(2)$. Then $\mathfrak{g} \times_{\mathfrak{h}/W} \mathfrak{h}$ is a double cover of $\mathfrak{g}$ branched over the nilpotent cone $xy + z^2 = 0$, so it's the surface $t^2 = xy + z^2$, a homogeneous ~~quadratic~~ Q in $\mathbb{C}^4$.

Also, $\tilde{\mathfrak{g}}$ in this case is the total space of $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ over $\mathbb{P}^1$ (exercise).

So $IH^j(Q) = \begin{cases} \mathbb{C}, & j = 0 \text{ or } j = 2 \\ 0 & \text{otherwise} \end{cases}$ while $H^j(Q) = \begin{cases} \mathbb{C}, & j = 0 \\ 0 & \text{otherwise} \end{cases}$