lecture 17.

The derived category of holonomic $\mathcal{D}$-modules.

Recall that the dual module $\mathbb{D}M$ is defined if M is coherent, and then so $\mathbb{D}^2M=M$. let us define for $\pi: X\to Y$ the functor $\pi_!$ by the formula $\pi_! M = \mathbb{D}\pi_*\mathbb{D}M$. Note that it's only partially defined, even if M is coherent (namely, $\pi_*\mathbb{D}M$ should be coherent).

If $\pi_! M$ is defined then there is a canonical morphism $\pi_! M \to \pi_* M$, which is defined as follows. First decompose π into a composition of an open embedding and a projective morphism. If $j: U\hookrightarrow X$ is an open embedding then $\forall$ $\mathcal{D}$-module N we have $Hom(N, j_*M) = Hom(N|_U, M)$, So $Hom(j_!M, j_*M) = Hom((j_!M)|_U, M) = Hom(M,M)$, so the identity map $M\to M$ defines a morphism $j_!M \longrightarrow j_*M$.

But for a projective morphism $\pi_* = \pi_!$

Similarly, we can define functors

$\pi^*_M = \mathbb{D}\pi^!\mathbb{D}M$ (defined if $\pi^!\mathbb{D}M$ is coherent).

<u>Theorem</u>. Let $\pi: X \to Y$ be a smooth morphism.

(1) $\pi^!$ maps $\mathcal{D}_{coh}(\mathcal{D}_Y)$ to $\mathcal{D}_{coh}(\mathcal{D}_X)$

(2) $\pi^* = \mathbb{D}\pi^!\mathbb{D} = \pi^![2(\dim Y - \dim X)]$

(3) π^* is left adjoint to π_*.

E.g. in the case of open embedding, $\pi^* = \pi^!$

<u>Example</u>. Let Y be a point. Then $\pi^! k = \mathcal{O}_X[\dim X]$, and
$\pi^* k = \mathbb{D}\pi^!\mathbb{D}k = \mathbb{D}\pi^! k = \mathbb{D}\,\mathcal{O}_X[\dim X]$
$= \mathcal{O}_X[-\dim X] = \pi^! k[-2\dim X]$. More generally if $X = W \times Y$, W smooth
$\pi^* M = M \boxtimes \mathcal{O}_W[\dim W]$
$\pi^! M = M \boxtimes \mathcal{O}_W[-\dim W]$.

We will skip the proof.

Of course, it would be nice to have a situation when $\pi_!$ and π^* are guaranteed to exist. In fact, the following theorem shows that it happens for holonomic $\mathcal{D}$-modules.

<u>Theorem</u>. $\mathcal{D}_{hol}(\mathcal{D}_X)$ is stable under π_*, $\pi^!$, $\mathbb{D}$, and $\boxtimes$.

Corollary On $\mathcal{D}_{hol}$ we can always define $\pi_! = \mathbb{D}\pi_*\mathbb{D}$ and $\pi^* = \mathbb{D}\pi^!\mathbb{D}$. Defined this way $\pi_!$ is left adjoint to $\pi^!$ ~~and right adjoint to π^*.~~ ~~(the fact that $\pi_!$ is right adjoint to π^* comes from dualizing the statement that π^* is left adjoint to $\pi^!$)~~

~~π^* is left adjoint to π_*~~

Proposition 1) π^* is left adjoint to π_*

2) $\pi^!$ is right adjoint to $\pi_!$

~~3) π^* [illegible]~~

Proof. 2) is obtained from (1) by dualization. To prove (1), it suffices to prove it for an open embedding and for projective morphism. For projective morphism, $\pi_* = \pi_!$, so we are done. For open embedding $\pi^* = \pi^!$, so again we are done by dualization.

Thus, on $\mathcal{D}_{hol}$ we have "formalism of 6 functors" $\pi^*, \pi^!, \pi_*, \pi_!, \boxtimes, \mathbb{D}$.

Before proving the theorem, let's give an example of application. Namely, let us classify irreducible holonomic $\mathcal{D}$-modules

<u>Lemma</u>. Let M be a holonomic on X. Then there is an open set $U \subset X$ such that for $j: U \to X$, $j^! M$ is $\mathcal{O}$-coherent.

<u>Pf.</u> We know $j^! U$ is $\mathcal{O}$-coherent iff $U \cap SS(M) =$ zero section of T^*U. Since $SS(M)$ is conic, it follows that every fiber of $SS(M) \to X$ is either 0 or of dimension ≥ 1. Since $\dim SS(M) = \dim X$, the fiber over generic point must be 0. The set U of such points satisfies the conditions of the lemma). $\blacksquare$

It is clear that we can choose U in the lemma to be affine.

Now consider the case when M is irreducible. In this case let's restore M from $j^! M$. First of all, if M is irreducible, so is $j^! M$. Indeed, suppose we have a short exact sequence

$0 \to K \to j^! M \to N \to 0$

With $K, N \neq 0$. Then we have a map ~~Then we have as~~

$M \to j_* N$ (note that if U is affine, j_* is exact).

Let $\tilde{K}$ be the kernel of this map. Then $j^! \tilde{K} = K$, so $\tilde{K}$ is nonzero but different from M. This contradicts the irreducibility of M.

Now let $N = j^! M$ and we want to restore M. In fact, we will see that we can do more: to every holonomic module N on U we can canonically attach a new module $j_{!*} N$ on X (call the minimal or intermediate or Goresky-MacPherson-Deligne) extension of N, which in particular will solve our problem in the case when N is irreducible.

Remark. Note that if M, M' are irreducible and $j^! M \cong j^! M'$ then $M \cong M'$. Indeed: $\varphi \overset{\neq 0}{\in} \mathrm{Hom}(j^! M, j^! M')$ can be viewed as $\tilde{\varphi} \in \mathrm{Hom}(M, j_* j^! M')$, such that $\tilde{\varphi} \neq 0$. However, we have a short exact sequence $0 \to M' \to j_* j^! M' \to i_* i^! M'[1] \to 0$, where $i : Z \to X$, $Z = X \setminus U$ divisor.

As $i_* i^! M'[1]$ is supported on Z, $\tilde{\varphi}(M)$ lands in M'. So $\tilde{\varphi} : M \simeq M'$

The intermediate extension $j_{!*}N$ is uniquely characterized by the following theorem.

<u>Theorem</u>. Let X be irreducible and $U \subset X$ open subset. For every holonomic $\mathcal{D}_U$-module N there exists a unique $\mathcal{D}_X$-module M satisfying the following properties:

1) $j^! j_{!*}(N) = N$ (so $j_{!*}N$ is an extension of N to X).

2) $j_{!*}$ has no submodules or quotients concentrated on $X \setminus U$.

We claim that if N is irreducible then $j_{!*}N$ is also irreducible. Indeed, assume not. Clearly, all composition factors of $j_{!*}N$ ~~should~~ except one should be concentrated on $X \setminus U$, otherwise they will have nonzero restr. to U. So if there are more than one comp. factor, it will be a sub a a quotient.

In particular, we see that any irreducible ~~$\mathcal{D}_U$~~ N has a unique irreducible extension to X, namely the intermediate extension $j_{!*}N$.

<u>Construction of $j_{!*}N$</u>

As suggested by the notation, it should somehow be constructed from $j_!N$ and j_*N.

We have $j_*: D^{\geq 0}_{hol}(\mathcal{D}_U) \to D^{\geq 0}_{hol}(\mathcal{D}_X)$, (cohomology in ≥ 0 degrees)

$j_!: D^{\leq 0}_{hol}(\mathcal{D}_U) \longrightarrow D^{\leq 0}_{hol}(\mathcal{D}_X)$ - (cohomology in ≤ 0 degrees)

Consider the map $j_!N \to j_*N$ that we constructed in the beginning of the lecture. By the remark above $j_!N \in \mathcal{D}^{\leq 0}_{hol}$, $j_*N \in \mathcal{D}^{\geq 0}_{hol}$, so the map factorizes as

$$j_!N \longrightarrow H^0(j_!N) \longrightarrow H^0(j_*N) \longrightarrow j_*N$$

Let $j_{!*}N$ be the image of this map. Note that if j is an affine embedding then $j_{!*}N$ is simply the image of $j_!N$ in j_*N (both functors $j_!$, j_* are exact in this case).

We claim that $j_{!*}N$ defined in this way satisfies property (1) and (2). First of all, the map $j_!N \to j_*N$

restricts to the identity on U, so property (1) is satisfied.

Let us now show that property (2) is satisfied. We have maps $j_! N \to j_{!*}N \to j_* N$. It's easy to see that for any $\mathcal{D}_X$-module L, the map $Hom(j_{!*}N, L) \to Hom(j_! N, L)$ is injective (since $j_! N \to j_{!*}N$ is surjective). Similarly $Hom(L, j_{!*}N) \to Hom(L, j_* N)$ is injective.

Now suppose L is supported on Z. Then $Hom(L, j_* N) = Hom(j^! L, N) = 0$, and $Hom(j_! N, L) = Hom(N, j^! L) = 0$. Thus, $Hom(L, j_{!*}N) = Hom(j_{!*}N, L) = 0$ and $j_{!*}N$ satisfies cond. (2). This proves the theorem.

<u>Example</u>. Let $\underset{\mathbb{A}^1 \setminus 0}{U} \subset \mathbb{A}^1$, $j : U \hookrightarrow \mathbb{A}^1$. Then $j_* \mathcal{O} = k[x, x^{-1}]$ (extension by poles) - extension of δ by $\mathcal{O}$, so $j_! \mathcal{O} = \mathbb{D} j_* \mathbb{D} \mathcal{O}$ - extension of $\mathcal{O}$ by δ. So $j_{!*}\mathcal{O} = \mathcal{O}$. This is $\langle \log x \rangle / \mathcal{O}$.

However, consider a $\mathcal{D}$-module $N = \langle \log x \rangle_U$ on U (extension of $\mathcal{O}$ by $\mathcal{O}$). Then $j_{!*}N = \langle \log x \rangle_{\mathbb{A}^1}$, with composition series $\mathcal{O}\,\delta\,\mathcal{O}$. Indeed, this $\mathcal{D}$-module has no submodules and quotients supported on $\{0\}$.

So we see that for reducible N, $j_{!*}N$ may contain composition factors supported on $Z = X \setminus U$ (in the middle of the composition series). Also the sequence $0 \to j_{!*}\mathcal{O} \to j_{!*}N \to j_{!*}\mathcal{O} \to 0$ isn't exact in the middle term, so the functor $j_{!*}$ is neither left nor right exact (so does not make sense in the derived category).

Let us now show that properties (1) and (2) define $j_{!*}N$ uniquely. Let M be another $\mathcal{D}$-module satisfying conditions 1 and 2. So $j^!M = j^*M = N$. So by adjointness, the map $j_!N \to j_*N$ factorizes through the sequence $j_!N \xrightarrow{\alpha} M \xrightarrow{\beta} j_*N$.

These maps become isomorphisms when restricted to U.
Thus Coker α is a quotient of M concentrated on $X \setminus U$, hence 0.
So α is surjective. Similarly, ker β is a submodule of M concentrated on $X \setminus U$, hence β is injective. Thus $M = j_{!*} N$.

Note that $j_{!*}$ is defined for a locally closed embedding.

__Lemma__. $j_{!*}$ commutes with $\mathbb{D}$.

__Proof__. It's clear that $\mathbb{D} j_{!*} N$ satisfies the axioms of $j_{!*} N$ (and recall that $\mathbb{D} N$ is a D-module if N is holonomic (i.e. not just a complex), since it's so for affine space.

Here is one of the most important applications of this construction (probably one of the most important ones). Let X be a singular variety. Let $U \subset X$ be a smooth dense open set. Set $IC_X = j_{!*} \mathcal{O}_U \in \mathcal{M}_{hol}(\mathcal{D}_X)$.
We call IC_X the Intersection cohomology D-module of X.

Let $\pi : X \to pt$, then $\pi_* IC(X)$ is called the Intersection cohomology of X. If X is smooth then $IC_X = \mathcal{O}_X$, so Intersection cohomology = ordinary cohomology. (We make an appropriate shift in defining the intersection cohomology).

Assume that X is projective. Then $\mathbb{D}$ commutes with π_*, so we have

$$\mathbb{D}\pi_*(IC_X) = \pi_* \mathbb{D}(IC_X) = \pi_*(IC_X),$$

so Intersection cohomology satisfies Poincare duality (even for singular X). This is a generalization of Poincare duality for ordinary cohomology.

Description of irreducible holonomic $\mathcal{D}$-modules.

It's clear from the above discussion that irreducible holonomic $\mathcal{D}$-modules on X are essentially classified by pairs (Z, N) where $Z \subset X$ is a smooth locally closed subvariety of X and N is an irreducible $\mathcal{O}$-coherent $\mathcal{D}$-module on ~~some open subset $U \subset X$~~ Z.

Namely, the corresponding $\mathcal{D}$-module is $j_{!*}N$, where $j: Z \to X$ is a natural embedding. Indeed, let M be an irreducible $\mathcal{D}_X$-module, and let $Y \subset X$ be the support of M. This is an irreducible closed subset of X, and by Kashiwara, M corresponds to some irreducible $\mathcal{D}_Y$-module M_Y. We now choose Z to be any smooth open subset of Y on which M is $\mathcal{O}$-coherent.

Moreover, if $Z' \subset Z$ is open, then $(Z', N|_{Z'})$ and (Z, N) correspond to the same holonomic $\mathcal{D}$-module holonomic module on X. In this case we'll say that (Z, N) and (Z', N'), $N' = N|_{Z'}$, are equivalent. Then irreducible holonomic $\mathcal{D}$-modules correspond to equiv. classes of pairs (Z, N).