To show that $R\pi_{*\,sheaf}(\cdot\cdot)$ is quasicoherent we will use the following general theorem:

<u>Theorem</u>. Let $\mathcal{A}$ be a quasicoherent sheaf of associative algebras on X. Then

$D^b_{qcoh}(\mathcal{M}(\mathcal{A})) \cong D^b(\mathcal{M}_{qcoh}(\mathcal{A}))$, where the first category is the full subcategory of $D^b(\mathcal{M}(\mathcal{A}))$ consisting of complexes of quasicoherent cohomology.

(When $\mathcal{A} = \mathcal{O}$, this is a homework problem)

Using a decomposition of π into a locally closed embedding and projection, one can show that $\pi_* M$ has quasicoherent cohomologies.

There is also another proof, based on explicit construction of direct image. Let K° be a complex of quasicoherent $\mathcal{D}$-modules representing $M \overset{L}{\underset{D_X}{\otimes}} D_{X\to Y}$. Let us take a cover $X = \bigcup_i U_i$ of X by affine open sets. Consider the Čech complex of K° corresponding to this cover, i.e. the total complex of the following bicomplex:

$$\bigoplus_{\alpha} (j_\alpha)_* (K^\bullet)|_{U_\alpha} \to \bigoplus_{\alpha_1,\alpha_2} (j_{\alpha_1,\alpha_2})_* (K^\bullet)|_{U_{\alpha_1}\cap U_{\alpha_2}} \to \cdots$$

Where $j_{\alpha_1,\ldots,\alpha_m} : U_{\alpha_1} \cap \cdots \cap U_{\alpha_m} \to X$

Then $K^\bullet$ and $\check{C}(K^\bullet)$ are quasiisomorphic, and moreover $R\pi_\cdot(K^\bullet) = \pi_\cdot \check{C}(K^\bullet)$.

So $\pi_*(M) = R\pi_\cdot(K^\bullet)$ consists of quasicoherent modules.

It remains to construct a complex $K^\bullet$ of quasicoherent modules representing $M \otimes^L_{\mathcal{D}_X} \mathcal{D}_{X\to Y}$. For this purpose, consider the Koszul complex $Kos(M) = M \otimes_{\mathcal{O}_X} dR(\mathcal{D}_X) \otimes_{\mathcal{O}_X} K_X$

Since $dR(\mathcal{D}_X)$ is a locally free resolution of K_X as a right $\mathcal{D}_X$-module, $dR(\mathcal{D}_X) \otimes_{\mathcal{O}_X} K_X^{-1}$ is a locally free resolution of $\mathcal{O}_X$ as a left $\mathcal{D}_X$-module. (recall that $dR(\mathcal{D}_X)$ is

$$0 \to \mathcal{D}_X \to \Omega^1 \otimes_{\mathcal{O}} \mathcal{D}_X \to \cdots \to \Omega^n \otimes_{\mathcal{O}} \mathcal{D}_X \to 0).$$

~~Also~~ Members of the complex $Kos(M)$ are locally free in the ∂-direction, and such objects form a class

adapted to $\underset{\mathcal{D}_X}{\otimes} \mathcal{D}_{X\to Y}$,
since $\mathcal{D}_{X\to Y}$ is locally free over $\mathcal{O}_X$
(this means that if we choose coordinates $x_1, \dots, x_n$, $Kos(M)$ consists of free $k[\partial_1, \dots, \partial_n]$-modules). Thus $K^{\cdot}$ is quasicoherent and represents $M \overset{L}{\underset{\mathcal{D}_X}{\otimes}} \mathcal{D}_{X\to Y}$, as desired. ∎

To compute direct images, one uses that any morphism $\pi: X \to Y$ is a composition of a locally closed embedding and a projection. Namely, since we agreed that X is quasiprojective, we have a locally closed embedding $i: X \to \mathbb{P}^N$, so we have

$$\begin{tikzcd} X \arrow[r, "i\times\pi"] \arrow[rr, bend right, "\pi"'] & \mathbb{P}^n \times Y \arrow[r, "p"] & Y. \end{tikzcd}$$

Here are some special cases.

1). If $j: X \to Y$ is an open embedding then direct image for $\mathcal{D}_X$-modules coincides with the usual (derived) direct image of $\mathcal{O}_X$-modules (as $\mathcal{D}_{X\to Y} = \mathcal{O}_X \underset{\mathcal{O}_Y}{\otimes} \mathcal{D}_Y = \mathcal{D}_X$ in this case), so $j_* M = Rj_{*\,sheaf}(M)$.

(2) If $\overset{\pi}{\overset{\shortparallel}{i}}: X \to Y$ is a closed embedding the the functor π_{*sheaf} is exact, and the functor $\underset{\mathcal{D}_X}{\otimes} \mathcal{D}_{X\to Y}$ is also exact, so π_* is exact and $\pi_* M = \pi_{*sheaf}(M \underset{\mathcal{D}_X}{\otimes}$ ~~essentially it's~~ (locally it's the functor we defined before for affine varieties).

(3) Let $\pi: X \to Y$ be smooth (i.e. a flat morphism with smooth fibers). In this case $M \overset{L}{\underset{D_X}{\otimes}} D_{X\to Y}$ is represented by the relative de Rham complex

$$0 \to M \to M \underset{\mathcal{O}}{\otimes} \Omega^1_{X/Y} \to \cdots \to M \otimes \Omega^{\dim X - \dim Y}_{X/Y} \to 0,$$

where $\Omega^i_{X/Y}$ are fiberwise differential forms. (We discussed such an example when we discussed hypergeometric functions). So $\pi_*(M) = R\pi_{*sheaf}(dR_{X/Y}(M))$, i.e. if $M = \mathcal{O}$, it is ~~generically~~ a bundle with a flat connection whose fibers are the de Rham cohomology of the fibers of π, and the connection is the Gauss-Manin connection.

Base change property.

Suppose we have a morphism $\pi: X \to Y$ and $\tau: S \to Y$. Then we can pull back π to S via τ, and get a diagram

$$\begin{tikzcd} X \times_Y S \arrow[r, "\tilde{\tau}"] \arrow[d, "\tilde{\pi}"] & X \arrow[d, "\pi"] \\ S \arrow[r, "\tau"] & Y \end{tikzcd}$$

For any $\mathcal{F} \in \mathcal{D}^b(\mathcal{D}_X)$, we have $\tau^! \pi_* \mathcal{F}$ and $\tilde{\pi}_* \tilde{\tau}^! \mathcal{F}$.

<u>Theorem.</u> We have a natural isomorphism of functors $\tau^! \pi_* \mathcal{F} \cong \tilde{\pi}_* \tilde{\tau}^! \mathcal{F}$.

<u>Proof.</u> Let's decompose τ as a locally closed embedding and a projection.

(1) for open embedding, statement is clear, since direct image is compatible to restriction to open subsets.

(2) Let τ be a projection. So $S = Y \times Z$ (can take $Z = \mathbb{P}^n$), $X \times_Y S = X \times Z$.

We have the following diagram

$$\begin{tikzcd} X \times Z \arrow[r, "\tilde{\tau}"] \arrow[d, "\tilde{\pi}"] & X \arrow[d, "\pi"] \\ Y \times Z \arrow[r, "\tau"] & Y \end{tikzcd}$$

We have $\quad \tilde{\tau}^! \mathcal{F} = \mathcal{F} \boxtimes \mathcal{O}_Z [\dim Z]$

and $\quad \tilde{\pi}_* \tilde{\tau}^! (\mathcal{F}) = \pi_* \mathcal{F} \otimes \mathcal{O}_Z [\dim Z] = \tau^! \pi_* \mathcal{F}$.

(3) Suppose τ is a closed embedding $(\tau = i)$
Consider the following diagram:

$$\begin{tikzcd} W \arrow[r, hook, "\tilde{i}"] \arrow[d, "\pi|_W"] & X \arrow[d, "\pi"] & U \arrow[l, hook', "\tilde{j}"'] \arrow[d, "\pi|_U"] \\ S \arrow[r, hook, "i"] & Y & V \arrow[l, hook', "j"] \end{tikzcd}$$

Here S is closed in Y and V is the complement of S, $W = \pi^{-1}(S)$, $U = \pi^{-1}(V)$.

We have an exact triangle $\quad (\forall \mathcal{F}$ on $X)$

$$\tilde{i}_* \tilde{i}^! \mathcal{F} \longrightarrow \mathcal{F} \longrightarrow \tilde{j}_* \tilde{j}^! \mathcal{F} \quad \text{(see appendix,}$$

Which gives rise to the following diagram

$$\begin{tikzcd} \pi_* \tilde{i}_* \tilde{i}^! \mathcal{F} \arrow[r] & \pi_* \mathcal{F} \arrow[r] \arrow[d, equal] & \pi_* \tilde{j}_* \tilde{j}^! \mathcal{F} \arrow[d, equal, "\text{by (1).}"] \\ i_* i^! \pi_* \mathcal{F} \arrow[r] & \pi_* \mathcal{F} \arrow[r] & j_* j^! \pi_* \mathcal{F} \end{tikzcd}$$

We need to construct an isomorphism between two left terms of this diagram.
Note that if we lived in an abelian category, and we had short exact

sequences on the rows, we'd get such an isomorphism automatically. But in the derived category we don't because of the non-canonicity of the cone.

However, it follows from the 5-lemma that if we construct any morphism that makes the diagram commutative then it is an isomorphism (as it's an isomorphism in cohomology, so a quasi isomorphism)

This can be done in the following way. Let $R \in \mathcal{D}_S(\mathcal{D}_Y)$, $T \in \mathcal{D}(\mathcal{D}_V)$. (here $\mathcal{D}_S(\mathcal{D}_Y)$ is the category of $\mathcal{D}$-modules on Y set-theoretically supported on S.

Then $Hom(R, j_* T) = Hom(j^! R, T) = 0$

Now take $R = \pi_* \tilde{i}_* \tilde{i}^! F$. Then from the above exact triang. we have

$$0 = Hom(R, j_* \pi^* \tilde{j}^! F[-1]) \to Hom(R, i_* i^! \pi_* F) \to$$
$$\to Hom(R, \pi_* F) \to Hom(R, j_* \pi^* \tilde{j}^! F) = 0.$$

Hence, $Hom(R, i_* i^! \pi_* F) = Hom(R, \pi_* F)$.

Since we have a canonical element of $Hom(R, \pi_* F)$, we also get an element of $Hom(R, i_* i^! \pi_* F)$. The fact that it makes

the diagram commutative is clear.
One can show that this isomorphism is independent on the decomposition of a morphism into a locally closed embedding and projection.

<u>More Properties of π_* and $\pi^!$</u>

<u>Lemma.</u> 1) $\pi^!$ maps coherent modules to coherent for smooth maps (in this case $\pi^!$ is exact up to shift).
2) π_* for projective morphism maps coherent modules to coherent ones.

<u>Pf.</u> 1) is clear from the definition. (reduces to affine case).
2) Note that for $\pi: X \to pt$, X smooth, $\pi(\mathcal{D}_X) = \mathcal{O}_X$, so for X affine statement is false. To prove the statement, note that it is true for closed embeddings, so it's enough to prove it for $X = \mathbb{P}^N \times Y$, $\pi: \mathbb{P}^N \times Y \to Y$ the projection to 2nd component, and Y is affine. In the category of coherent $\mathcal{D}$-modules $\mathcal{D}_X$ is a projective generator (since $\mathbb{P}^N$ is $\mathcal{D}$-affine), so it's enough to prove the statement for $\mathcal{D}_X$.
We have $\mathcal{D}_{X \to Y} = \mathcal{O}_{\mathbb{P}^N} \times \mathcal{D}_Y$. Hence

$$(\Omega_X \underset{\mathcal{O}_X}{\otimes} \mathcal{D}_X) \underset{\mathcal{D}_X}{\otimes} \mathcal{D}_{X\to Y} = \left((\Omega_{\mathbb{P}^N} \boxtimes \Omega_Y) \underset{\mathcal{O}_{\mathbb{P}^N} \boxtimes \mathcal{O}_Y}{\otimes} \mathcal{D}_{\mathbb{P}^N} \boxtimes \mathcal{D}_Y\right)$$

$$\underset{\mathcal{D}_X = \mathcal{D}_{\mathbb{P}^N \times \mathcal{O}_Y}}{\otimes} \mathcal{O}_{\mathbb{P}^N} \boxtimes \mathcal{D}_Y = \Omega_{\mathbb{P}^N} \boxtimes (\Omega_Y \underset{\mathcal{O}_Y}{\otimes} \mathcal{D}_Y)$$

Thus $\pi_* \mathcal{D}_X = \mathcal{D}_Y[-N]$ (as $R\pi_{*\,sheaf}(\Omega_{\mathbb{P}^N}) = \mathbb{C}[-N]$). This implies the statement. ∎

<u>Prop.</u> Let $\pi: X \to Y$ be projective. Then

1) π_* is left adjoint to $\pi^!$

2) π_* commutes with $\mathbb{D}$.

<u>Pf.</u> It's enough to assume that $X = \mathbb{P}^N \times Y$, where Y is affine (as for closed embedding we already know this). Let us prove (1). Let $\mathcal{F} \in \mathcal{D}(\mathcal{D}_X)$ and $\mathcal{G} \in \mathcal{D}(\mathcal{D}_Y)$. We need to show that $R\mathrm{Hom}(\pi_* \mathcal{F}, \mathcal{G}) \cong R\mathrm{Hom}(\mathcal{F}, \pi^! \mathcal{G})$. It's enough to consider $\mathcal{F} = \mathcal{D}_X$ since $\mathcal{D}_X$ is a projective generator and $\mathcal{D}(\mathcal{D}_X)$ is equivalent to the homotopy category of free complexes (we'll also need to check compatibility with morphisms).

As we computed, $\pi_* \mathcal{D}_X = \mathcal{D}_Y[-N]$.

So $\mathrm{Hom}(\pi_* \mathcal{D}_X, \mathcal{G}) \cong \mathrm{Hom}(\mathcal{D}_Y[-N], \mathcal{G}) = R\Gamma(Y, \mathcal{G})[N]$

On the other hand, $\pi^! \mathcal{G} = \mathcal{O}_{\mathbb{P}^N} \boxtimes \mathcal{G}[N]$, so

$$\mathrm{Hom}(\mathcal{D}_X, \pi^! \mathcal{G}) = R\Gamma(X, \pi^! \mathcal{G})[N] = R\Gamma(Y, \mathcal{G})[N],$$

as desired.

To prove the second statement, we again can construct the above isom. for D_X, which is done by a similar calculation.

<u>Theorem</u>. Let $\pi : X \to Y$ be smooth. Then

1) $\mathbb{D}\pi^! [\dim Y - \dim X] = \pi^! \mathbb{D} [\dim X - \dim Y]$.

2) $\pi^! [2(\dim X - \dim Y)]$ is left adjoint to π_*.

<u>Proof</u>. Exercise.

Appendix to Lecture 16.

Exact triangle attached to a closed embedding.

Let X be a topological space, $Z \subset X$ a closed subset. Let $\mathcal{F}$ be a sheaf on X. We say that a section $s \in \Gamma(U, \mathcal{F})$ ($U \subset X$ open) is supported on Z if $s|_{U \setminus Z} = 0$.

Let $j : X \setminus Z \hookrightarrow X$ be the open embedding and let $j^! : Sh(X) \to Sh(X \setminus Z)$ be the corresponding restriction functor. Also let j_* be the associated direct image functor $Sh(X \setminus Z) \to Sh(X)$. We have a natural exact sequence

$$0 \to \mathcal{F}_Z \to \mathcal{F} \to j_* j^! \mathcal{F}$$

of sheaves on X, where $\mathcal{F}_Z$ is the subsheaf of $\mathcal{F}$ of sections supported on Z, i.e. $\forall U \subset X$ $\Gamma(U, \mathcal{F}_Z) = \{ s \in \Gamma(U, \mathcal{F}) \mid s \text{ is supported on } Z\}$.

Moreover, if $\mathcal{F}$ is injective, the morphism $\mathcal{F} \to j_* j^! \mathcal{F}$ is surjective (because injective sheaves are flabby, i.e. admit extension of sections). So for injective sheaves we have $0 \to \mathcal{F}_Z \to \mathcal{F} \to j_* j^! \mathcal{F} \to 0$.

This means that for any complex of sheaves $\mathcal{F}$, we have an exact triangle in the derived category of sheaves:

$$0 \to R\mathcal{F}_Z \to \mathcal{F} \to Rj_{*\,\text{sh}}\, j^! \mathcal{F} \to 0$$

Now, assume that X is a smooth variety, $Z \subset X$ a (smooth) closed subvariety, and $\mathcal{F}$ is a quasi-coherent sheaf. Then note that $\Gamma(\mathcal{F}_Z)$ is the sections of $\mathcal{F}$ set-theoretically supported on Z. (i.e. killed by some power of the ideal of Z).

Now let $\mathcal{F}$ be a $\mathcal{D}$-module (right). Then note that sections of $i_* i^{!0}\mathcal{F}$ are the sections of $\mathcal{F}$ set-theoretically supported on Z, and $i^{!0}$ is ~~right~~ left exact (as we showed before). Hence in the derived category of $\mathcal{D}$-modules we have an exact triangle

$$0 \to i_* i^! \mathcal{F} \to \mathcal{F} \to j_* j^! \mathcal{F} \to 0$$

where $\mathcal{F}$ is a complex of $\mathcal{D}$-modules, and $i^!$ stands for $Ri^{!0}$, while $j_* = Rj_{*\text{sheaf}}$.