Lecture 15.

Derived functors.

In the previous lecture we gave a definition of the derived category $\mathcal{D}(\mathcal{A})$ of an abelian category $\mathcal{A}$ and described its properties. The motivation for this was to show that under suitable conditions, any left (or right) exact functor $F: \mathcal{A} \to \mathcal{B}$ between abelian categories gives rise to the derived functor

$$RF: \mathcal{D}^+(\mathcal{A}) \to \mathcal{D}^+(\mathcal{B}), \text{ resp } LF: \mathcal{D}^-(\mathcal{A}) \to \mathcal{D}^-(\mathcal{B}),$$

which is exact. Let us do this now.

Suppose that $F: \mathcal{A} \to \mathcal{B}$ is a left exact functor. We would like to define the derived functor $RF: \mathcal{D}^+(\mathcal{A}) \to \mathcal{D}^+(\mathcal{B})$ by universal properties. Let $K^+(F)$ be the natural extension of F to the functor $K^+(\mathcal{A}) \to K^+(\mathcal{B})$.

<u>Def.</u> The derived functor of F is a pair (RF, ε_F), where $RF: \mathcal{D}^+(\mathcal{A}) \to \mathcal{D}^+(\mathcal{B})$ is an exact functor, and $\varepsilon_F: Q_B \circ K^+(F) \to RF \circ Q_A$ is a morphism of functors satisfying the following universality condition: for every exact functor $G: \mathcal{D}^+(\mathcal{A}) \to \mathcal{D}^+(\mathcal{B})$ and a morphism of functors $\varepsilon: Q_{\mathcal{B}} \circ K^+(F) \to G \circ Q_A$ there exists a unique morphism $\eta: RF \to G$ for which the following diagram is commutative:

$$\begin{tikzcd}[column sep=small] & Q_B \circ K^+(F) \arrow[dl, "\varepsilon"'] \arrow[dr, "\varepsilon_F"] & \\ G \circ Q_A & & RF \circ Q_A \arrow[ll, "\eta \circ Q_A"'] \end{tikzcd}$$

Here $Q_A : K^+(\mathcal{A}) \to \mathcal{D}^+(\mathcal{A})$, $Q_B : K^+(\mathcal{B}) \to \mathcal{D}^+(\mathcal{B})$.

The same def. for right exact functors and LF, with $+ \Leftrightarrow -$.

It's easy to see from this definition that if RF exists, it is unique, but we have to show that it exists and learn to compute it. For this we need the notion of adapted class of objects.

<u>Def.</u> A class $\mathcal{R}$ of objects of $\mathcal{A}$ is adapted to F if it's stable under finite direct sums and

(1) F maps exact complexes of objects of $\mathcal{R}$ to exact complexes

and

(2) Any object of $\mathcal{A}$ is a subobject of an object of $\mathcal{R}$.

<u>Example.</u> if F is left exact ~~th~~ and $\mathcal{A}$ has enough injectives then the class of injectives is adapted to F.

A similar definition applies to right exact functors.

Namely, for an adapted class we require that any object of $\mathcal{A}$ is a quotient of one from $\mathcal{R}$.

Now assume that $\cancel{R}F$ is left exact, and $\mathcal{R}$ is a class of objects adapted to F. Then we can define RF as follows. For any $X \in \mathcal{D}^+(\mathcal{A})$, we can choose $K \in C^+(\mathcal{R})$ which represents X, and then we set $RF(X) = Q_{\beta}(F(K))$.

One can show that it is independent on the choice of K.

<u>Example</u> If $X \in \mathcal{A}$, then K can be taken to be any resolution of X by objects of $\mathcal{R}$:

$$0 \to R_0 \to R_1 \to R_2 \to \cdots$$

(such that $H^0 \cong X$, $H^i = 0$ for $i > 0$).

Such a resolution exists by definition.

More generally, ~~assume that~~ K exists because of the following proposition:

<u>Prop.</u> If $\mathcal{R}$ is a class of objects adjusted to F, and $S_{\mathcal{R}}$ - class of

quasiisomorphisms in $\mathcal{R}$. Then $S_{\mathcal{R}}$ is a localizing class and $\mathcal{K}^{\pm}(\mathcal{R})[S_{\mathcal{R}}^{-1}] \to \mathcal{D}^{\pm}(\mathcal{A})$ is an equivalence.

In particular, every $X \in \mathcal{D}^{\pm}(\mathcal{A})$ can be represented by $K \in K^{\pm}(\mathcal{R})$ (here we use $+$ for right exact functors and $-$ for left exact functors).

Prop. 1. $H^n(RF(K^\bullet))$ is a subquotient of $\bigoplus_{p+q=n} R^pF(H^q(K^\bullet))$ (in fact, have a spectral seq.).

2. Let $\mathcal{A} \xrightarrow{F} \mathcal{B} \xrightarrow{G} \mathcal{C}$, both left exact. If there are admissible classes $R_{\mathcal{A}}, R_{\mathcal{B}}$ for F, G such that $F(R_{\mathcal{A}}) \subset R_{\mathcal{B}}$, then $R(G \circ F) = RG \circ RF$.

The same can be done for right exact functors.

Ex. 1) If A a commutative ring, $F(N) = M \otimes_A N$, for fixed M, then $LF = \otimes^L_A N$. This functor maps $\mathcal{D}^-$ to $\mathcal{D}^-$. (adjusted objects: flat or projective; adapted)

2) The derived functor of Hom is $R\mathrm{Hom}: \mathcal{D}^+ \to \mathcal{D}^+$. (adjusted objects: injective / projective).

It can be shown that these functors don't depend on which argumente we derive.

Now let us apply this Machinery to D-modules. We will write $D(\mathcal{D}_X)$ for $D^b(\mathcal{M}(\mathcal{D}_X))$ (with appropriate ℓ and r subscripts).

<u>Theorem.</u> If X is smooth then $\mathcal{D}^b(\mathcal{M}_{hol}(\mathcal{D}_X)) \cong \mathcal{D}^b_{hol}(\mathcal{D}_X)$, where the last category is the full subcategory of $\mathcal{D}^b(\mathcal{D}_X)$, which consists of objects represented by complexes with holonomic cohomology. In particular, if M, N are holonomic D-modules, then $\mathrm{Ext}^i_{\mathcal{M}(\mathcal{D}_X)}(M,N) \cong \mathrm{Ext}^i_{\mathcal{M}_{hol}(\mathcal{D}_X)}(M,N)$

<u>Remark.</u> Note that if $\mathcal{A} \subset \mathcal{B}$ is a Serre subcategory, it is not true in general that $\mathrm{Ext}^i_{\mathcal{A}}(M,N) \cong \mathrm{Ext}^i_{\mathcal{B}}(M,N)$ for $M, N \in \mathcal{A}$. E.g. Take $\mathcal{A}$ = f.d. $\mathfrak{sl}_2$-mod. $\mathcal{B}$ = all $\mathfrak{sl}_2$-mod. Then $\mathrm{Ext}^3_{f.d.}(\mathbb{C},\mathbb{C}) = 0$, $\mathrm{Ext}^3_{all}(\mathbb{C},\mathbb{C}) = \mathbb{C}$.

We will now study the bounded derived category of $\mathcal{D}$-modules. We will drop the superscript b and write $\mathcal{D}^l(\mathcal{D}_X)$, $\mathcal{D}^r(\mathcal{D}_X)$ for the categories of left and right $\mathcal{D}$-modules. ~~As~~

Assume X is smooth. Let's define duality $\mathbb{D} : \mathcal{D}^l_{coh}(\mathcal{D}_X) \to \mathcal{D}^r_{coh}(\mathcal{D}_X)$ by $M \mapsto \underline{RHom}(M, \mathcal{D}_X)$, for any coherent left $\mathcal{D}_X$-module. Here $\underline{RHom}(M, \mathcal{D}_X)$ is a quasicoherent sheaf of right $\mathcal{D}_X$-modules whose sections on every affine open $U \subset X$ are $\Gamma(U, \underline{RHom}(M, \mathcal{D}_X)) = Hom_{\mathcal{D}}(\Gamma(U, M), \mathcal{D}_U)$. The functor $\underline{Hom}$ is left exact, so we can consider the derived functor (adjusted objects are locally free $\mathcal{D}_X$-modules).

Let us define the functor

$\mathbb{D} : \mathcal{D}^l(\mathcal{D}_X) \to \mathcal{D}^l(\mathcal{D}_X)$ by $\mathbb{D}(M) =$
$= R\underline{Hom}(M, \mathcal{D}_X \otimes \Omega_X^{-1})[n]$, where $n = \dim X$.

<u>Theorem</u>. 1) $\mathbb{D}^2 = Id$ on $\mathcal{D}^b(\mathcal{M}^l_{coh}(\mathcal{D}_X))$.

2) Let $M, N \in \mathcal{D}^b(\mathcal{M}^l_{coh}(\mathcal{D}_X))$. Then
$Hom(M, \mathbb{D}(N)) \cong Hom(N, \mathbb{D}(M))$.

Proof. (1) Let R be the class of locally free $\mathcal{D}_X$-modules. This class is adapted to $\mathbb{D}$, so it's enough to check this for complexes of locally free locally f.g. $\mathcal{D}_X$-modules. For such modules,

$\underline{Hom}(\underline{Hom}(M, \mathcal{D}_X), \mathcal{D}_X) \cong M$. So we have

So we have a natural morphism $Id \to \mathbb{D}^2$. Since it's an isomorphism for every object of R, it's an isomorphism.

(Remark: shift by n is in order to make sure that if X is affine then $\mathbb{D}$ maps D-modules to D-modules)

The second statement follows from the fact that $Hom(M, Hom(N, D_X)) \cong Hom_{\mathcal{D}_X}(N, Hom_{\mathcal{D}_X}(M, D_X))$

$\cong Hom_{\mathcal{D}_X\text{-bimod}}(M \boxtimes N, \mathcal{D}_X)$

Theorem. $RHom(M, N) = RHom(\mathbb{D}(N), \mathbb{D}(M))$ for any coherent left $\mathcal{D}_X$-modules $M, N \in D^b(\mathcal{M}^{\ell}_{coh}(\mathcal{D}_X))$

Pf. There is a morphism

$RHom(M, N) \to RHom(\mathbb{D}(N), \mathbb{D}(M))$

which is an ~~eqiv~~ isomorphism since $\mathbb{D}$ is an equivalence.

<u>Inverse image.</u> Let $\pi : X \to Y$ be a morphism of smooth irred. varieties and π^*_{sheaf} the sheaf-theoretic inverse image (for q.coherent sheaves). Define the inverse image functor $\pi^!$ by $\pi^!(M) = \mathcal{O}_X \overset{L}{\otimes}_{\pi^*_{sheaf}\mathcal{O}_Y} \pi^*_{sheaf}(M)[\dim X - \dim Y]$

If π is a closed embedding then $\pi^!$ is the left derived functor of $\pi^{!0}$, hence the notation. (recall: $\pi^{!0} = L^{\dim X - \dim Y}\pi^{*0}$, $\pi^{*0} = R^{\dim Y - \dim X}_{L}\pi^!$)

As before, define

$$\mathcal{D}_{X\to Y} = \pi^! \mathcal{D}_Y[\dim Y - \dim X] = \mathcal{O}_X \otimes_{\pi^*_{sheaf}\mathcal{O}_Y} \pi^*_{sheaf}\mathcal{D}_Y,$$

<u>Direct image.</u> Let $\pi : X \to Y$ be any morphism. Let $M \in D(\mathcal{D}_X)$. Define

$$\pi_*(M) = R\pi_{*sheaf}(M \overset{L}{\otimes}_{D_X} D_{X\to Y})$$

By definition, this is a complex of sheaves of $\mathcal{D}_Y$-modules, and it's not clear why they are quasicoherent (as we have to resolve by injective sheaves etc.) but we will show it.

<u>Example.</u> Let Y be a point, X a smooth variety. Then $D_{X\to Y} = \mathcal{O}_X$, so we get

$\pi_* M = R\Gamma\left(M \overset{L}{\underset{D_X}{\otimes}} \mathcal{O}_X\right)$. Thus $\pi_* M$ is the hypercohomology of the De Rham complex of M, which, as was shown by Grothendieck, is the cohomology of X for $M = \mathcal{O}_X$ (living between $-\dim X$ and $\dim X$).

~~Ex. $X = \mathbb{P}^1$, $M \neq \mathcal{O}_X$.~~

Appendix (to lecture 15)

Proposition. Let $0 \to K^0 \to K^1 \to \cdots$ be a complex in an abelian category $\mathcal{A}$ with enough injectives. Then $\exists$ an embedding $i : K^\bullet \to I^\bullet$ of K into a complex of injectives.

Proof. We can embed $K^0 \hookrightarrow I^0$.

Now suppose we have

$$
\begin{tikzcd}
0 \arrow[r] & K^0 \arrow[r, "d_0"] \arrow[d, "i_0"] & K^1 \\
0 \arrow[r] & I^0 &
\end{tikzcd}
$$

image of $(i_0, -d_0)$

Let $M_1 = (I^0 \oplus K^1)/K^0$, and embed $M_1 \hookrightarrow I^1$.

Then we have

$$
\begin{tikzcd}
0 \arrow[r] & K^0 \arrow[r, "d_0"] \arrow[d, "i_0"] & K^1 \arrow[d, "i_1"] \\
0 \arrow[r] & I^0 \arrow[r, "\tilde{d}_0"] & I^1
\end{tikzcd}
$$

and note that i_1 is injective.

Now consider

$$
\begin{tikzcd}
0 \arrow[r] & K^0 \arrow[r, "d_0"] \arrow[d, "i_0"] & K^1 \arrow[r, "d_1"] \arrow[d, "i_1^*"] & K^2 \\
0 \arrow[r] & I^0 \arrow[r, "d_1"] & I^1 &
\end{tikzcd}
$$

image of $(i_1, -d_1)$

and let $M_2 = I^1 \oplus K^2 / K^1 + d_1(I_0)$.

Embed $M_2 \overset{i_2}{\hookrightarrow} I^2$. Then we get

$$
\begin{tikzcd}
0 \arrow[r] & K^0 \arrow[r, "d_0"] \arrow[d, "i_0"] & K^1 \arrow[r, "d_1"] \arrow[d, "i_1"] & K^2 \arrow[d, "i_2"] \\
0 \arrow[r] & I^0 \arrow[r, "\tilde{d}_0"] & I^1 \arrow[r, "\tilde{d}_1"] & I^2
\end{tikzcd}
$$

Note that i_2 is injective. indeed, assuming $\mathcal{A} \subset A\text{-m}$ suppose $i_2(k_2) = 0$. Then

$$(0, k_2) = (i_1(k_1), -d_1(k_1)) + (\tilde{d}_0(v_0), 0)$$

where $v_0 \in I_0$. So $\tilde{d}_0(v_0) = i_1(k_1)$

i.e. $v_0 = i_0(k_0)$, $k_1 = d_0(k_0)$, so

$$k_2 = -d_1(k_1) = -d_1 d_0(v_0) = 0.$$

And we continue in this way ∎

The same argument is used if instead of injectives we use any class $\mathcal{R}$ of objects such that any object of $\mathcal{A}$ is a subobject of an object of $\mathcal{R}$. ∎

Cor. Any complex in $K^+(\mathcal{A})$ is quasiisomorphic to a complex of objects of $\mathcal{R}$ (if $\mathcal{R}$ is additive).

Proof. $K^\circ \hookrightarrow I_1^\circ$, embed cokernel into I_2°, ... Then take total complex of

$$I_1^\circ \to I_2^\circ \to \cdots$$