D-modules Nov. 7 (Roman)

Derived categories

Last time: start with an abelian category $\mathcal{A}$. define $D(\mathcal{A})$ as $\mathcal{C}(\mathcal{A})$

derived category complexes

with inverted quasi-isomorphisms

Basic features: $\mathcal{A} \subset D(\mathcal{A})$ full subcat

$\subset \mathcal{C}(\mathcal{A})$

"H^0 complexes"

shift functors acting on $\mathcal{C}(\mathcal{A})$

$C^i[n] = C^{i+n}$

Note: $D(\mathcal{A}) \supset \mathcal{A} \ni M$ $\quad \underset{n>0}{M[n]}$ is in deg $-n$

$\underset{n \geq 0}{M[n]} \in D^{\leq 0} = \{C \in D(\mathcal{A}),\ H^i(C) \neq 0 \Rightarrow i \leq 0\}$

$D^{\geq 0} = \{C \in D(\mathcal{A}) \mid H^i(C) \neq 0 \Rightarrow i \geq 0\} \supset M[-n],\ n \geq 0$

$\mathrm{Hom}_{D(\mathcal{A})}(A, B[n]) = \mathrm{Ext}^n(A,B) \qquad A,B \in \mathcal{A}$

Main construction: derived functors

$F: \mathcal{A} \to \mathcal{B}$ is a functor between abelian categories which is either left/right exact

$0 \to A \to B \to C \to 0$ s.e.s

$\Downarrow$

$0 \to F(A) \to F(B) \to F(C)$ is exact

left exactness e.g. • $\Gamma: Sh \to Vect$
• invariance

$0 \to A \to B \to C \to 0 \Rightarrow F(A) \to F(B) \to F(C) \to 0$

is right exact e.g. • $\otimes$ of modules
• coinvariance

under weak assumptions can extend it

to left exact - right derived functor RF

right exact - left —"— LF

Then $R^i F(M) := H^i RF(M)$

$L^{-i} F(M) := H^{-i} LF(M)$ - non-zero for $i \geq 0$ only

$R^0 F = F$

$0 \to F(A) \to F(B) \to F(C) \to R^1 F(A) \to$
$\to R^1 F(B) \to \ldots$

$\to L^i F(A) \to L^i F(B) \to L^i F(C) \to L^{i+1} F(A) \to$

Ex: • $\mathcal{A} = Sh(X)$ ab. grp. $\quad R^i\Gamma(\mathcal{F}) = H^i(X, \mathcal{F})$

• $\mathcal{A}$ - any $\quad F : \mathcal{A} \to Ab.$

$X \to Hom(M, X)$, $\quad R^iF(M) = Ext^i(M, X)$

$\mathcal{A} \ni M$ fixed

• $\mathcal{A}$ - R-mod, unital ring $\quad$ M - fixed right module

$F : X \longmapsto M \otimes_R X$

$L^{-n}F(X) = Tor_n^R(M, X)$

Recall

$f : K^\bullet \to L^\bullet$ defined $C(f)^\bullet$ - new complex

$C(f)^i = K^{i+1} \oplus L^i$,

$d(k^{i+1}, l^i) = (-d_K k^{i+1}, f(k^{i+1}) + d_L(l^i))$

We have a s.e.s of complexes

$0 \to L^\bullet \to C(f) \to K^\bullet[1] \to 0$

which termwise splits & any termwise split sequence has this form

Recall that for a s.e.s of complexes get a long exact sequence of coh.

$$H^i(K^\cdot) \xrightarrow{f} H^i(L^\cdot) \to H^i(C(f)) \to H^{i+1}(K) \to \cdots$$

If $F: \mathcal{A} \to B$ is just additive, then

$$0 \to F(L) \to F(C(f)) \to F(K) \to 0$$

is also a split s.e.s of complexes also yielding a long exact sequence

$$H^i(F(K^\cdot)) \to \cdots$$

Exercise 0) The canonical splitting is a map of complexes iff $f = 0$

1) There exists a splitting iff $f \sim 0$ ← homotopic

i.e. $\exists\, h: L^\cdot \to K[-1]^\cdot$

$$L^i \to K^{i-1}$$

$$f = dh + hd$$

Def: A distinguished triangle in $D(\mathcal{A})$ is the data of

$$X \to Y \to Z \to X[1]$$

s.t.

$\exists\ K_x \xrightarrow{f} K_y \to C(f) \to K[1]$

which induces $X \to Y \to Z \to X[1]$

Ex: Given such a dist. triang., the triang.

$$Y \xrightarrow{g} Z \xrightarrow{h} X[1] \xrightarrow{-f[1]} Y[1]$$

is also disting. (Hint: use Cyl)

Recall:

A functor $F: D(A) \to D(B)$ is exact or triangulated if F sends dist. triang to dist. triang.

RF, LF are like that

Using the definition of $D(A)$ as a formal localization - hard to describe maps in part to show A is additive

Similar story: R - non-comm. ring $S \subset R$ invert S. A nice description exists under Ore condition $as^{-1} \sim t^{-1}b$

complexes in $\mathcal{A}$

The category $\mathcal{C}(\mathcal{A})$ can be enriched

For $K^\cdot, L^\cdot \in \mathcal{C}(\mathcal{A})$ can define a complex of abelian groups.

$\mathrm{Hom}^\cdot(K, L)$: $\quad \mathrm{Hom}^i(K^\cdot, L^\cdot) = \prod_n \mathrm{Hom}_{\mathcal{A}}(K^n, L^{n+i})$

$$d_{\mathrm{Hom}} h \big|_{K^i} = d_L h + (-1)^{i+1} h d_K$$

Notice

$\mathrm{Hom}(K^\cdot, L^\cdot) = \{ h \in \mathrm{Hom}^\cdot(K^\cdot, L^\cdot) \mid dh = 0 \}$

can also consider

$$H^0 \mathrm{Hom}^\cdot(K, L) = \frac{\{ \varphi : K^\cdot \to L^\cdot \mid \text{commuting with } d \}}{\{ \varphi \mid \varphi = dh + hd \}}$$

for some $h : L^\cdot \to K^\cdot[-1]$

$\underbrace{\varphi \sim 0}$ homotopic

$\mathcal{H}_0(\mathcal{A})$; $\mathrm{Ob}(\mathcal{H}_0(\mathcal{A})) = \mathrm{Ob}(\mathcal{C}(\mathcal{A}))$

homotopy category

$\mathrm{Hom}_{\mathcal{H}_0}(K^\cdot, L^\cdot) := H^0 \mathrm{Hom}^\cdot(K^\cdot, L^\cdot)$

If $\varphi = dh + hd$, φ induces 0 on H^*

$c \in K^i$, $dc = 0$, $\varphi(c) = \underbrace{dh(c)}_{\sim 0 \text{ in } H^i(L^\cdot)} + h\cancel{d}(c)$

The functor $\mathcal{C}(\mathcal{A}) \to D(\mathcal{A})$ factors through

$$\mathcal{H}_0(\mathcal{A}) \longrightarrow D(\mathcal{A})$$

<u>Rem</u>: can define distinguished triangles in $\mathcal{H}_0(\mathcal{A})$

a triangle

$X \to Y \to Z \to X[1]$ is dist. if it comes from $K \xrightarrow{f} L \to C(f) \longrightarrow K[1]$

Will construct $D(\mathcal{A})$ with a triang. functor

$$\mathcal{H}_0(\mathcal{A}) \longrightarrow D(\mathcal{A})$$

<u>$D(\mathcal{A}) \leftsquigarrow \mathcal{H}(\mathcal{A})$</u> inverting quasi-iso.

<u>Claim</u>: The set S of quasi-iso. in $\mathcal{H}_0(\mathcal{A})$ form a localizing class.

$s, t \in S \Rightarrow st \in S$

$$\begin{tikzcd}
W \arrow[r, dashed, "g"] \arrow[d, dashed, "s \Rightarrow t"'] & Z \arrow[d, "s"] \\
X \arrow[r, "t"'] & Y
\end{tikzcd}
\qquad
\begin{tikzcd}
W & Z \arrow[l, "g"'] \\
X \arrow[u, dashed, "s \Rightarrow t"] & Y \arrow[l, "t"] \arrow[u, "s"']
\end{tikzcd}$$

$s : X' \longrightarrow X$

$f, g : X \longrightarrow Y$, $\exists s$ $sf = sg$

$\Leftrightarrow \exists t : Y \to Y'$, $ft = gt$

$Ob(S^{-1}\mathcal{K}_0(\mathcal{A})) = Ob(\mathcal{K}_0(\mathcal{A}))$

$\mathrm{Hom}(X, Y)$:

$$\begin{tikzcd} & X' \arrow[dl, "s"'] \arrow[dr] & \\ X & & Y \end{tikzcd}$$

$f\!f$ $\exists$

$$\begin{tikzcd} & & & X''' \arrow[dl, dashed, "r"'] \arrow[dr, dashed, "h"] & & & \\ & & X' \arrow[dl, "s"'] \arrow[dr, "f"] & \sim & X'' \arrow[dl, "t"'] \arrow[dr, "g"] & & \\ & X & & Y \quad X & & Y & \end{tikzcd}$$

$C(\mathcal{A}) \to \mathcal{K}_0(\mathcal{A})$ is a triangulated cat

$\cup$ $\qquad\qquad$ $\cup$

$\mathcal{A}$cyclic $\qquad$ $\mathcal{A}$cyclic$(\mathcal{A})$

$\parallel$

$\{C^\cdot \mid H^i(C^\cdot) = 0 \ \forall i\}$

closed under direct summands

$X \to Y \to Z \to X[1]$ distinguished, $X, Y \in \mathcal{A}$cycl $\Rightarrow$ Z is $\mathcal{A}$cycl

$D(\mathcal{A}) = \mathcal{K}_0(\mathcal{A}) / Acycl(\mathcal{A})$

Fact: $\mathcal{C} \to \mathcal{C}/\mathcal{D}$

$\mathrm{Hom}_{\mathcal{C}}(X,Y) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}/\mathcal{D}}(\bar{X},\bar{Y})$

$\mathcal{C}$ - triang cat, $\mathcal{D}$ - thick subcat.

If $X \in {}^{\perp}\mathcal{D}$ or $Y \in \mathcal{D}^{\perp}$

${}^{\perp}\mathcal{D} = \{X \mid \mathrm{Hom}(X,D) = 0 \ \forall D \in \mathcal{D}\}$

Exercise: suppose that $I^{\bullet}$ is a complex of injective modules bounded below. Then $I^{\bullet} \in Acycl^{\perp}$

$P^{\bullet}$ is a complex of projective modules bounded above (on the right). Then $P^{\bullet} \in {}^{\perp}Acycl$.

So can compute $\mathrm{Hom}_{D(\mathcal{A})}$ using projective/injective resolutions

Cor: $\mathrm{Hom}_{\mathcal{K}_0}(P^{\bullet}, C^{\bullet}) \xrightarrow{\sim} \mathrm{Hom}_{D(\mathcal{A})}(P^{\bullet}, C^{\bullet})$,

$\mathrm{Hom}_{\mathcal{K}_0}(C^{\bullet}, I^{\bullet}) \simeq \mathrm{Hom}_{D(\mathcal{A})}(C^{\bullet}, I^{\bullet})$

Now if $F : A \to B$ is a left derived functor.

$$\begin{tikzcd} RF : D(A) \arrow[r] & D(B) \\ \mathcal{H}_0(A) \arrow[u] \arrow[r, "F"'] \arrow[ur, phantom, "\cancel{\circlearrowleft}"] & \mathcal{H}_0(B) \arrow[u] \end{tikzcd}$$

But it commutes on $Acycl^{\perp}$, in part for complexes of injective bounded below.

LF for a right exact F $\Rightarrow$ can be computed on bounded above proj. complexes