Derived categories.

Let $\mathcal{A}$ be an abelian category and $\mathcal{C}(\mathcal{A})$ be the category of all complexes over $\mathcal{A}$. Let $\mathcal{C}^+(\mathcal{A})$ be the category of complexes K with $K^i=0$ for $i \ll 0$, and $\mathcal{C}^-(\mathcal{A})$ be the category of complexes (differential goes from left to right) with $K^i=0$ for $i \gg 0$. Let $\mathcal{C}^b(\mathcal{A})$ be the intersection of these, i.e. the category of bounded complexes. Let $\mathcal{C}_0^o(\mathcal{A})$ be the category of complexes with zero differential. We have a functor of cohomology $H: \mathcal{C}(\mathcal{A}) \to \mathcal{C}_0(\mathcal{A}) = \bigoplus_{i\in\mathbb{Z}} \mathcal{A}$, which attaches to every complex its cohomology.

Recall that $f: C \to D$ is a quasi isomorphism if $H(f): H(C) \to H(D)$ is an isomorphism.

<u>Theorem 1.</u> There exists a unique up to a canonical equivalence category $\mathcal{D}(\mathcal{A})$ (called the derived category) together with a functor $Q: \mathcal{C}(\mathcal{A}) \to \mathcal{D}(\mathcal{A})$ such that

1) if $f: K^\cdot \to L^\cdot$ is a quasiisomorphism

then $Q(f)$ is an isomorphism;

2) The pair is universal for this property, i.e. for any functor $F: \mathcal{C}(\mathcal{A}) \to \mathcal{D}'$ satisfying 1), there is a unique functor $G: \mathcal{D}(\mathcal{A}) \to \mathcal{D}'$ such that $F = G \circ Q$.

The objects of $\mathcal{D}(\mathcal{A})$ are the objects of $\mathcal{C}(\mathcal{A})$.

<u>Remark</u>. More generally, if $\mathcal{C}$ is any category and S is any class of morphisms, then we can define the category $\mathcal{C}[S^{-1}]$ and a functor $\mathcal{C} \to \mathcal{C}[S^{-1}]$ which satisfy condition 1) and 2) above: $\mathcal{D}(\mathcal{A}) = \mathcal{C}(\mathcal{A})[S^{-1}]$ where S is the class of quasi isomorphisms.

<u>Motivation</u>: Suppose $\mathcal{A} = A\text{-mod}$. Then any $M \in \mathcal{A}$ has a projective resolution, and any two resolutions are homotopy equivalent: there is a pair of morphisms

$$\begin{tikzcd} P' \arrow[dr] \arrow[d, shift right, "g"'] & \\ P'' \arrow[u, shift right, "f"'] \arrow[r] & M \end{tikzcd}$$

such that $f \circ g$ and $g \circ f$ are homotopic to the identity, i.e. $f g - 1 = h d + d h$, etc.

We want all these resolutions to be one object, defined canonically. In particular, we want this to be the case for $M = 0$, i.e. we want any exact complex of projectives to be zero.

Structure of derived categories.

1. Shift functor : $K^\circ \to K^\circ[i]$, $K[i]^j = K^{j+i}$.
2. Distinguished triangles.

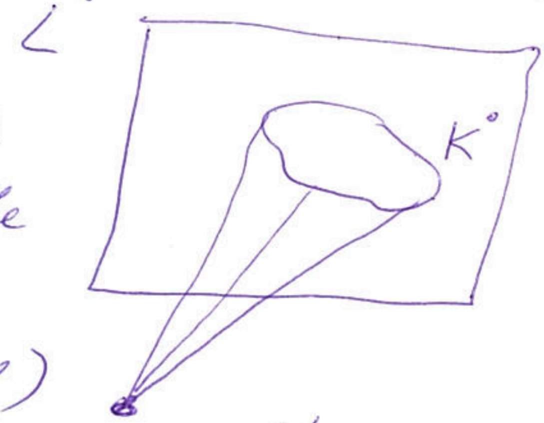
For a left exact functor $F: \mathcal{A} \to \mathcal{B}$ we'll define $RF: \mathcal{D}(\mathcal{A}) \to \mathcal{D}(\mathcal{B})$, and for a right exact functor $G: \mathcal{A} \to \mathcal{B}$ we'll define $LG: \mathcal{D}(\mathcal{A}) \to \mathcal{D}(\mathcal{B})$ and we will want to say that it ~~maps~~ ~~short~~ is exact. So we'll need an analogy of short exact sequences. But we don't have short exact sequences in the usual sense, since the category $\mathcal{D}(\mathcal{A})$ is not abelian, and we don't have the notion of kernel and cokernel of morphisms. So we have to replace these notions by something.

For this we will define the notion of the cone of a morphism of complexes. Let $f: K^\circ \to L^\circ$ be a morphism of complexes. Define the complex $C(f)^\circ$, called the cone of f, as follows:

$C(f)^\circ = K^\circ[1] \oplus L^\circ$, i.e. $C(f)^i = K^{i+1} \oplus L^i$;

with differential $d(k^{i+1}, l^i) = (-d_K k^{i+1}, f(k^{i+1}) + d_L l^i)$.

<u>Remark</u>. Suppose that K° and L° come from actual simplicial complexes, and (as complexes, computes reduced cohomology) we have a simplicial map $K^\circ \to L^\circ$. Then we can consider the space obtained by gluing a cone over K to L along the map f. Then the complex $C^\circ(f)$ computes the (reduced) homology of this space. So the cohomology of $C^\circ(f)$ is the reduced cohomology of the space obtained by contracting the image of f to a point (if f is an inclusion).



<u>Exercise</u>. Show that if f is an embedding, $C^\circ(f)$ is isomorphic to L°/K°.

<u>Lemma</u>. The sequence

$$\cdots \to H^i(K) \to H^i(L) \to H^i(C(f)) \to H^{i+1}(K) \to \cdots$$

is exact (where the connection map corresponds to the projection $C(f) \to K[1]$

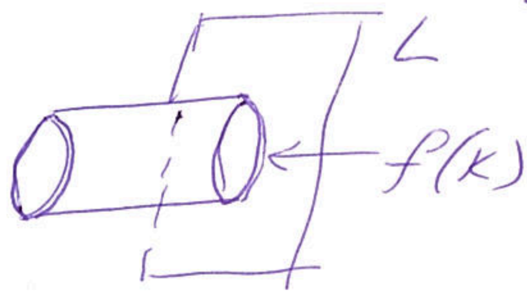
<u>Proof</u>. If $f: K^\circ \hookrightarrow L^\circ$, then $C(f) \simeq L^\circ/K^\circ$ (quasiisomorphic), so this is the well known long exact sequence.

More generally, we define Cylinder of f:

$Cyl(f) = K^\circ \oplus K^\circ[1] \oplus L^\circ$ with

$$d(k^i, k^{i+1}, l^i) = (d_K k^i - k^{i+1}, -d_K k^{i+1}, f(k^{i+1}) - d_L l^i).$$

The picture for it:



The natural inclusion $L^\circ \hookrightarrow Cyl(f)$ is a quasiisomorphism, and the sequence $K^\circ \to Cyl(f) \to C(f)$ is an exact sequence of complexes.

So we get a long exact sequence, as desired.

<u>Def</u>. A distinguished triangle in $D(\mathcal{A})$ is a triangle $X \to Y \to Z \to X[1]$ which is the image under Q of

$K^\circ \to L^\circ \to C(f) \to K[1]$

<u>Main problem</u>: Cone of $f: X \to Y$, $X, Y \in D(\mathcal{A})$ is not canonically defined. It is unique, but up to a noncanonical isomorphism.

Because to construct the cone, we need to pick complexes representing X, Y.

Def. Let $\mathcal{A}$, $\mathcal{B}$ be categories. A functor $F: \mathcal{D}(\mathcal{A}) \to \mathcal{D}(\mathcal{B})$ is called exact if it commutes with shift and maps distinguished triangles to distinguished triangles.

Def. $X \in \mathcal{D}(\mathcal{A})$ is an H^0-complex if $H^i(X) = 0$ for $i \neq 0$.

Lemma. The inclusion $\mathcal{A} \hookrightarrow \mathcal{D}(\mathcal{A})$ induces an equivalence between $\mathcal{A}$ and the ~~derived~~ full sub category of H^0-complexes.

Let $X, Y \in \mathcal{A}$. Then we can define Ext by $\mathrm{Ext}^i(X,Y) = \mathrm{Hom}_{\mathcal{D}(\mathcal{A})}(X^\circ, Y^\circ[i])$ (note that $\mathrm{Hom}_{\mathcal{A}}(X,Y) = \mathrm{Hom}_{\mathcal{D}(\mathcal{A})}(X^\circ, Y^\circ)$). If $\mathcal{A}$ has enough projectives or injectives, then ~~we~~ one can show that it is the same definition as before.

Another way of thinking about the derived category.

It is not hard to prove that $\mathcal{D}(\mathcal{A})$ exists and is unique, but not easy to understand what morphisms are.

<u>Ex.</u> Suppose $\mathcal{C}$ is a category with one object X and $\operatorname{End} X = A$, a monoid.
Let $\underset{\text{mult. subset}}{S \subset A}$. Then $A[S^{-1}]$ is the monoid consisting of words with letters $a \in A$ and $s^{-1}, s \in S$ with equiv. relations:

1) $\cdots \boxed{a_1}\boxed{a_2} = \cdots \boxed{a_1 a_2} \cdots$
2) $\cdots s_1^{-1} s_2^{-1} \cdots = (s_1 s_2)^{-1} \cdots$ ~~if $s_1 s_2 \in S$~~
3) $\cdots s^{-1} s \cdots = \cdots \quad \cdots s s^{-1} \cdots = \cdots$

So it's hard to understand what morphisms are. But things are better if we have "Ore condition", i.e. the class S is localizable.

<u>Def.</u> A mult. closed set S is localizable if

(1) $\forall\, s \in S$ ($s: Z \to Y$) and f ($f: X \to Y$) $\exists\, t \in S$ and g such that diagrams are commutative:

$$\begin{tikzcd} W \arrow[r, dashed, "g"] \arrow[d, "t \in S"'] & Z \arrow[d, "s"] \\ X \arrow[r, "f"'] & Y \end{tikzcd} \quad (\text{i.e. } s^{-1} f = g t^{-1})$$

$$\begin{tikzcd} W & Z \arrow[l, dashed, "g"'] \\ X \arrow[u, "t"] & Y \arrow[l, "f"] \arrow[u, "s"'] \end{tikzcd} \quad (\text{i.e. } f s^{-1} = t^{-1} g)$$

So a right fraction can be rewritten as a left fraction, and vice versa.

<u>and</u> (2) let $f: X \to Y$. Then $\exists\, s \in S$ $sf = sg \Leftrightarrow \exists\, t \in S$ $ft = gt$

(this will be a condition for f to equal g in the localization).

If S is localizable, then the localization $\mathcal{C}[S^{-1}]$ has a nice description.
Morphisms can be represented by "roofs" (left or right)

$$\begin{tikzcd} & X' \arrow[dl, "s"'] \arrow[dr, "f"] & \\ X & & Y \end{tikzcd}$$
(this is morally fs^{-1})

or

$$\begin{tikzcd} & Y' \arrow[dr, "s"] & \\ X \arrow[ur, "f"] & & Y \end{tikzcd}$$
(this is morally $s^{-1}f$)

Moreover, two diagrams

$$\begin{tikzcd} & X' \arrow[dl, "s"'] \arrow[dr, "f"] & \\ X & & Y \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & X'' \arrow[dl, "t"'] \arrow[dr, "g"] & \\ X & & Y \end{tikzcd}$$
define the same morphism if $\exists X'''$ and $r: X''' \to X'$, $h: X''' \to X''$, $r, h \in S$ such that the following diagram is commutative:

$$\begin{tikzcd} & X''' \arrow[dl, "r"'] \arrow[dr, "h"] & \\ X' \arrow[d, "s"'] \arrow[drr, "f" near start] & & X'' \arrow[d, "g"] \arrow[dll, "t"' near start] \\ X & & Y \end{tikzcd}$$

(i.e. to show that $fs^{-1} = gt^{-1}$, we find r, h such that $fr = gh$, $sr = th$, then $fs^{-1} = frr^{-1}s^{-1} = ghr^{-1}s^{-1} = ghh^{-1}t^{-1} = gt^{-1}$).

<u>Prop.</u> If S is a localizable class then $\mathcal{C}[S^{-1}]$ is the category with objects = objects of $\mathcal{C}$ and morphisms = roofs with the equivalence rel. as above.