Lecture 12. Equivariant $\mathcal{D}$-modules.

Let X be an (smooth) algebraic variety and G an affine algebraic group acting on X. Recall the notion of a G-equivariant quasicoherent sheaf on X. To make the definition, let

$m: G\times G \to G$ be the multiplication map, and $\rho: G\times X \to X$ be the action of G on X. Clearly, we have a commutative diagram

$$\begin{tikzcd} & G\times X \arrow[dr, "\rho"] & \\ G\times G\times X \arrow[ur, "m\times Id"] \arrow[dr, "Id\times\rho"'] & & X \\ & G\times X \arrow[ur, "\rho"'] & \end{tikzcd}$$

i.e. $\rho\circ(Id\times\rho) = \rho\circ(m\times Id)$.

<u>Definition</u>. A quasicoherent sheaf M on X is said to be G-equivariant if it is endowed with an isomorphism

$\varphi: \rho^*M \to \mathcal{O}_G \boxtimes M$ such that the following diagram commutes:

$$\begin{tikzcd} (Id\times\rho)^*\rho^*M \arrow[r, "(Id\times\rho)^*(\varphi)"] \arrow[d, equal] & (Id\times\rho)^*(\mathcal{O}_G\boxtimes M) = \mathcal{O}_G\boxtimes\rho^*M \arrow[dr, "Id\times\varphi"] & \\ (m\times Id)^*\rho^*M \arrow[r, "(m\times Id)^*(\varphi)"] & (m\times Id)^*(\mathcal{O}_G\boxtimes M) = m^*\mathcal{O}_G\boxtimes M \arrow[r, "\sim"', "\psi_G\times Id"] & \mathcal{O}_G\boxtimes\mathcal{O}_G\boxtimes M \end{tikzcd}$$

where $\psi_G : m^*\mathcal{O}_G \xrightarrow{\sim} \mathcal{O}_G\boxtimes\mathcal{O}_G = \mathcal{O}_{G\times G}$ is the canonical is

(this means that we have a compatible family $\varphi_g : M^g \to M$ which depend algebraically on g)

Now we extend this definition to $\mathcal{D}$-modules.

<u>Definition</u>. An ~~quasi equ~~ weakly G-equivariant $\mathcal{D}$-module on X is a $\mathcal{D}_X$-module M equipped with an equivariant structure as a quasi-coherent sheaf, such that φ <u>is an isomorphism of $\mathcal{D}_X$-modules</u>.

Note that a weakly equivariant $\mathcal{D}$-module carries two actions of the Lie algebra $\mathfrak{g} = Lie(G)$ on ~~global~~ sections on any open set $U \subset X$:

1) the intrinsic action ρ_1, coming from the map $\mathfrak{g} \to Vect\, X \subset \mathcal{D}_X$, defined by the G-action on X (it does not involve the map φ).

2) The action ρ_{φ} coming from equivariant structure. Namely, if $L \in \mathfrak{g}$ and $\gamma(t)$ is a ~~1-parameter~~ curve in G such that $\gamma(0) = 1$, $\gamma'(0) = L$ then for a local section v of M, $L \cdot v = \left.\frac{d}{dt}\right|_{t=0} \gamma(t) \cdot v$

In general, these two actions are not the same.

<u>Definition.</u> A weakly G-equivariant D-module M is called G-equivariant if the above two actions are the same.
Rem. This is equivalent to φ being a D-module isomorphism.

<u>Example.</u> Let X be a point. A D-module on X is just a vector space V. Then a weakly G-equivariant D-module on X is the same thing as an ~~weakly~~ equivariant O-module, so it is an ~~locally~~ locally algebraic representation of G. However, the intrinsic action on G on V is zero. So an equivariant D-module on X is an algebraic repr. of G such that $Lie(G)$ acts trivially, so it's an action of G/G_0 on V. (where $G_0 \subset G$ is the connected component of the identity)

<u>Lemma.</u> ~~Suppose that a D-module~~ Let $L \in \mathfrak{g}$. Then $\rho_*(L) - \rho_{\varphi *}(L)$ is an endomorphism of M.

<u>Proof.</u> The group G acts on $\mathcal{D}_X$ by $g \circ A = gAg^{-1}$. It's easy to check that $[\rho_*(L), A] = [\rho_{\varphi_*}(L), A] = \frac{d}{dt}\Big|_{t=0} \gamma(t) \circ A$.

Thus we get that

$\rho_{\varphi*}(L) = \rho_*(L) + \lambda(L)$, where $\lambda(L)$ commutes with diff. operators. Thus if $\mathrm{End}\,M = k$ (e.g. M is irreducible) then

$$[\rho_{\varphi_*}(L), \rho_{\varphi_*}(L')] = [\rho_*(L), \rho_*(L')] = \rho_*([L,L']),$$
$$\rho_{\varphi*}([L,L'])$$

i.e. $\rho_{\varphi*} = \rho_*$ on $[L,L']$, i.e. $\lambda([L,L'])=0$,

i.e. $\lambda : \mathfrak{g}/[\mathfrak{g},\mathfrak{g}] \to \cancel{\mathrm{End}\,M}\ k$ a character.

In particular, if $\mathfrak{g}$ is perfect (e.g. semisimple), any weakly equivariant $\mathcal{D}$-module is automatically equivariant (for connected G). Also if M is irreducible (or more generally $\mathrm{End}\,M = k$), then a weakly equivariant $\mathcal{D}$-module can be made equivariant by multiplying φ by a character of G.

<u>Remark</u>. If X is affine, $\mathrm{End}\,M = k$, then a weakly G-equivariant structure on M is just a locally algebraic action of G on M whose derivative differs from the intrinsic action by

a map $\lambda : \mathfrak{g}/[\mathfrak{g},\mathfrak{g}] \to k$

<u>Remark</u>. We see that equivariance of a $\mathcal{D}$-module for connected G, unlike that of $\mathcal{O}$-module, is a property and not a structure, since the action of G is determined by the action of $\mathfrak{g}$. Equivariance is just the condition that the $\mathfrak{g}$-action on M integrates to an algebraic action of G on M (for affine X). If X is not affine, "integrates" should be understood in the sense of the definition of an equivariant quasicoherent sheaf. In particular, we see that the category of G-equivariant $\mathcal{D}$-modules for connected G is a full subcategory of the category of all $\mathcal{D}$-modules.

<u>Example</u>. Let us classify $\mathbb{C}^*$-equivariant $\mathcal{D}$-modules on $\mathbb{C}$. These are precisely the $\mathcal{D}$-modules ~~whose~~ in which the Euler element $x\partial$ acts semisimply with integer eigenvalues. The examples we seen are: $\mathcal{O}$, δ_0, extension of $\mathcal{O}$ by δ_0, extension of δ_0 by $\mathcal{O}$.

let us show that any irreducible equivariant $\mathcal{D}$-module is either $\mathcal{O}$ or δ_0. To do so, let M be irreducible and let $v \in M$ be such that $x\partial v = nv$ for some $n \in \mathbb{Z}$. We know from homework that $\mathcal{D}/(x\partial - n)$ generates an extension of $\mathcal{O}$ by δ_0 if $n \geq 0$ and an extension of δ_0 by $\mathcal{O}$ otherwise. This implies the statement.

From this it's easy to see that the only indecomposable equivariant $\mathcal{D}$-modules are the four modules as above.

<u>Exercise</u>. 1. Let X be a variety with a free action of G, i.e. X is a G-bundle over $X/_G = Y$. Then if $\pi : X \to Y$ is the projection then $\pi^* : \mathcal{M}^\ell(\mathcal{D}_Y) \to \mathcal{M}^\ell(\mathcal{D}_X)$ is an equivalence of $\mathcal{M}^\ell(\mathcal{D}_Y)$ onto $\mathcal{M}^\ell_G(\mathcal{D}_X)$, the category of G-equivariant $\mathcal{D}$-modules on X.

2. Let L be a line bundle on Y, X - set of nonzero vectors in the total space, and $G = \mathbb{C}^*$ (so $Y = X/_{\mathbb{C}^*}$). Consider $\mathcal{M}^\ell_{\mathbb{C}^*}(\mathcal{D}_X)_m$ the category of weakly equivariant $\mathcal{D}$-modules where two Lie algebra actions differ by m-th power of the generating character of $\mathbb{C}^*$. Then

Also let $\mathcal{M}^\ell(\mathcal{D}_Y(L^{\otimes m}))$ be the category of modules over the twisted differential operators (acting on $L^{\otimes m}$). Then π^* is an equivalence $\mathcal{M}^\ell(\mathcal{D}_Y(L^{\otimes m})) \xrightarrow{\sim} \mathcal{M}^\ell_{\mathbb{C}^*}(\mathcal{D}_X)_m$.

Remark. The above applies to smooth X. If X is singular affine, we can G-equivariantly embed X into an affine space, and use Kashiwara's theorem. Also, the definition requiring that φ is an isomorphism of $\mathcal{D}$-modules works for singular X.

Theorem. Let X be affine, and G have finitely many orbits on X. Then any f.g. G-equivariant $\mathcal{D}_X$-module is holonomic.

Proof. By picking a G-invariant subspace E of generators, we can define a G-invariant good filtration on M. For $v \in E$, we have $\rho(a)v = av$ for any $a \in \mathfrak{g}$, so $\mathrm{gr}\, M$ is a quotient of $\mathcal{O}(T^*X)\otimes E / I$, where I is defined by the relation $\rho(a) = 0$, $a \in \mathfrak{g}$. Thus, the singular support of M is contained in the union of conormal bundles of G-orbits, i.e. is Lagrangian ∎.

Ex. Suppose $X = G/H$. Then a G-equivariant $\mathcal{D}$-module on X is the same thing as a $\pi_0(H)$-module.