Lecture 11.

D-modules on general varieties.

Let X be any smooth algebraic variety (for simplicity X quasiprojective; this does not affect the statements but simplifies the exposition). Then $\exists$ a unique sheaf $\mathcal{D}_X$ of differential operators on X such that for any affine open $U \subset X$, $\Gamma(U, \mathcal{D}_X) = \mathcal{D}_U$. This sheaf is a sheaf of $\mathcal{O}_X$-modules by both left and right multiplication. Moreover, as we have shown, this sheaf is quasicoherent with respect to both left and right $\mathcal{O}_X$-module structures. So we have the categories of left and right quasicoherent $\mathcal{D}_X$-modules, $\mathcal{M}^\ell(\mathcal{D}_X)$ and $\mathcal{M}^r(\mathcal{D}_X)$ ($\mathcal{D}_X$ itself belongs to either). And these two categories are equivalent via $M \longmapsto M \otimes \Omega^n(X)$, where $n = \dim X$.

Many statements that we proved in the affine case are of local nature, so they remain true in general.

For example, this is the case for Kashiwara's theorem. If X is singular, we can cover X by affine open U_i such that U_i can be embedded into a smooth variety. In this case, a $\mathcal{D}$-module on X is a collection $\{M_i\}$ of $\mathcal{D}$-modules on U_i, $\forall i$, such that for any i,j we have an isomorphism $\eta_{ij}: M_i|_{U_j\cap U_i} \xrightarrow{\sim} M_j|_{U_j\cap U_i}$ and $\eta_{jk}\circ\eta_{ij} = \eta_{ik}$, $\eta_{ji} = \eta_{ij}^{-1}$. (here $M_i|_{U_j\cap U_i}$ is the $*$-pullback).

Let X be smooth. Then $\mathrm{gr}\,\mathcal{D}_X = p_*\mathcal{O}(T^*X)$ where $p: T^*X \to X$ is the standard projection. In this case we also have ~~a functor~~ the notion of singular support of a (locally) f.g. M $M \mapsto s.s.(M) \subset T^*X$. By Gabber's theorem, $ss(M)$ is coisotropic. Also have singular cycle ~~given by~~ $SC(M)$ (of dimension $d(M)$) where $d(M)$ is the maximum of local dimensions.

Def. M is holonomic if $d(s.s(M)) = n$

It is easy to see by a local argument that any holonomic $\mathcal{D}$-module has finite length.

sheaf-theoretic

<u>Inverse and direct images for arbitrary varieties.</u>

Let $\pi : X \to Y$ and $N \in \mathcal{M}^l(\mathcal{D}_Y)$. Then we can define the inverse image $\pi^{*o} N = \mathcal{O}_X \otimes_{\pi^*\mathcal{O}_Y} \pi^* N$ – ($\pi^* N$ is a sheaf of $\pi^*\mathcal{O}_Y$-modules (well defined since $\pi^*\mathcal{O}_Y \hookrightarrow \mathcal{O}_X$). This is local, so analogous to affine case. But with direct image, the situation is worse. Namely, we may try to define $\mathcal{D}_{X \to Y} = \pi^{*o}\mathcal{D}_Y = \mathcal{O}_X \otimes_{\pi^*\mathcal{O}_Y} \pi^*\mathcal{D}_Y$. If we set $\pi_{*o} M = \pi_*(M \otimes_{\mathcal{D}_X} \mathcal{D}_{X \to Y})$ then this functor is neither left nor right exact, since $\otimes$ is right exact but π_* is left exact. Also we can check that this definition is not compatible with composition. In fact, we will see that direct image is correctly defined only in the derived category.

Ex: $\begin{array}{ccccc} X & \xrightarrow{\pi} & Y & \xrightarrow{\eta} & Z \\ \parallel & & \parallel & & \parallel \\ SL(2) & & SL(2)/B & & pt \\ & & \parallel & & \\ & & \mathbb{P}^1 & & \end{array}$

$\eta_{*0}\pi_{*0}\Omega = \mathbb{C} = H^3(X, \mathbb{C})$

But $\pi_{*0}\Omega = 0$ since $H^2(B) = 0$.

So direct image not compatible with composition.

But for closed embeddings, the def. is still good since in this case we can define it locally in Y. smooth

$\mathcal{D}$-affine varieties. Def. X is called $\mathcal{D}$-affine if $\Gamma(X, ?)$ is an equivalence between $\mathcal{M}^\ell(\mathcal{D}_X)$ and modules over $\mathcal{D}_X^{glob}$. (for $\mathcal{O}$, this is Serre's def. of affine var.)

Theorem 1. $\mathbb{P}^n$ is $\mathcal{D}$-affine.

Pf. X is $\mathcal{D}$-affine $\Leftrightarrow$ Γ is exact, and $\forall M \neq 0$, $\Gamma(X, M) \neq 0$. Indeed, if X is $\mathcal{D}$-affine then these properties hold. Conversely, if Γ is exact and $\Gamma(X, M) \neq 0$ for any $M \neq 0$ Then $\mathcal{D}_X$ is projective and a generator for $\mathcal{M}^\ell(\mathcal{D}_X)$, so result follows.

Now let's prove the theorem. $X=\mathbb{P}^n=\mathbb{P}(V)$. Let $\tilde{V}=V\setminus 0$, $j:\tilde{V}\hookrightarrow V$ inclusion, $\pi:\tilde{V}\to\mathbb{P}(V)$ projection. Let $M\in\mathcal{C}(\mathcal{D}_{\mathbb{P}^n})$. Then

$$\Gamma(\tilde{V},\pi^{*\circ}M)=\bigoplus_{k\geq 0}\Gamma(\mathbb{P}V, M\otimes\mathcal{O}(k)).$$

(recall that $\pi^{*\circ}M=\mathcal{O}(\tilde{V})\otimes_{\mathcal{O}(\mathbb{P}V)} M$ and $\mathcal{O}(\tilde{V})=\bigoplus_{k\geq 0}\mathcal{O}(k)$). Consider the Euler field $\mathcal{E}=\sum x_i\partial_i$. This field acts on $\Gamma(\mathbb{P}V, M\otimes\mathcal{O}(k))$ by mult. by k. So $\mathcal{E}$ acts with eig ≥ 0 and $\Gamma(\mathbb{P}^n,M)=\Gamma(\tilde{V},\pi^{*\circ}M)^{\mathcal{E}}$. (Since $\mathcal{E}$ acts semisimply, taking invariants does not influence exactness), so we need to show (and Γ does not kill anything) that $\Gamma(\tilde{V},\pi^{*\circ}\,?)$ is exact. Consider the sequence

$$M\longmapsto\pi^{*\circ}M\longmapsto\Gamma(\tilde{V},\pi^{*\circ}M).$$

The first functor is exact, but the second may not be. In fact, we can decompose the last arrow as comp. of two functors: take $j_{*\circ}$ and then take global sections on V, and nonexactness may be hidden only in $j_{*\circ}$. (so $\Gamma(\tilde{V},\pi^{*\circ}M)=\Gamma(V,j_{*\circ}\pi^{*\circ}M)$)

Explanations. 1. Inverse image. The definition means that $\pi^{*o}N(U)$ for $U \subset X$ open affine is given by $\pi^{*o}N(U) = \pi_U^{*o}(N)$, where $\pi_U : U \to Y$. So in definition of π^{*o}, it's enough to assume that X is affine. Also can assume Y affine by taking an open affine cover of Y. In other words, we pull it back as an $\mathcal{O}$-module.

2. Let's compute $\pi_{*o}(\Omega_X)$ for $\pi: X \to pt$ with the wrong definition. we get

$$\pi_{*o}\Omega_X = \pi_*(\Omega_X \underset{\mathcal{D}_X}{\otimes} \mathcal{O}_X) = \pi_*(\Omega_X^n/\mathrm{Im}\, d) = \pi_*(F)$$

where $F = \mathrm{Coker}\,(d : \Omega_X^{n-1} \to \Omega_X^n)$.

Ex. $X = \mathbb{P}^1$. $0 \to \mathcal{O}_X/\mathbb{C} \xrightarrow{d} \Omega^1 \to F \to 0$

$$0 \to H^0(\mathcal{O}_X/\mathbb{C}) \to H^0(\Omega^1) \to H^0(F) \to H^1(\mathcal{O}_X/\mathbb{C}) \to H^1(\Omega^1)$$

$H^0(\Omega^1) = 0$, $H^1(\Omega^1) = \mathbb{C}$, $H^1(\mathcal{O}_X/\mathbb{C}) = H^2(\mathbb{C}) = \mathbb{C}$ ...

Now, if $j: U \to X$ is an open embedding then j_{*0} is left exact, and all derived functors (as it involves taking are concentrated at 0 (recall $j_{*0} M = j_{*sheaf}(M \otimes_{\mathcal{D}_{V\setminus 0}} \mathcal{D}_V)$, and j_{*sheaf} involves taking global sections over preimages of open sets containing zero)

Consider $j: \tilde{V} = V \setminus 0 \to V$. For any short exact sequence of $\mathcal{D}_{\mathbb{P}V}$-modules

$$0 \to M_1 \to M_2 \to M_3 \to 0,$$

we have the long exact sequence

$$0 \to j_{*0}\pi^{*0}M_1 \to j_{*0}\pi^{*0}M_2 \to j_{*0}\pi^{*0}M_3 \to R^1 j_{*0}\pi^{*0}M_1 \to \dots$$

By Kashiwara's theorem, R^1 is a direct sum of δ_0-modules (since it is supported at 0). The eigenvalues of $\mathcal{E}$ on $\Gamma(V, \delta_0)$ are $-1, -2, -3, \dots$

On the other hand, the eigenvalues of $\mathcal{E}$ on $\Gamma(V, j_{*0}\pi^{*0}M_3)$ are nonnegative by the above considerations. Hence the connecting homomorphism in the above long exact sequence is 0, (as it's $\mathcal{E}$-invariant) and thus the functor $\Gamma(V, j_{*0}\pi^{*0}\,?)$ is exact

It remains to show that $\Gamma(\mathbb{P}^n, M) \neq 0$ if $M \neq 0$. There exists $k \geq 0$ such that $\Gamma(\mathbb{P}V, M \otimes \mathcal{O}(k)) \neq 0$. (Indeed, it suffices to take some coherent subsheaf $\overline{M} \subset M$, and use that it's true for any nonzero coherent sheaf). Hence $\exists\, m \neq 0$, $m \in \Gamma(\tilde{V}, \pi^{*\circ} M)$ such that $\mathcal{E} m = k m$. Since $\mathcal{E} \partial_i m = (k-1)\partial_i m$, if there exists i such that $\partial_i m \neq 0$ then $\exists\, l \in \Gamma(\tilde{V}, \pi^{*\circ} M)$ with eigenvalue $k-1$. If $\partial_i m = 0$ $\forall i$, then $\mathcal{E} m = 0$. Hence by induction in k, $\Gamma(\tilde{V}, \pi^{*\circ} M)^{\mathcal{E}} \neq 0$. ■

This theorem is really a special case of the localization theorem.

<u>Example.</u> Let $\dim V = 2$. Then we get that $\mathcal{M}(\mathcal{D}_{\mathbb{P}^1}) = \mathcal{D}_{\mathbb{P}^1}^{glob}$-mod. So let's compute $\mathcal{D}_{\mathbb{P}^1}^{glob}$.

Consider the map $\varphi: U(sl_2) \to \mathcal{D}_{\mathbb{P}^1}^{glob}$, defined by sending elements of sl_2 to the corresponding vector fields on $\mathbb{P}^1$.

<u>Thm.</u> φ is an isomorphism $U(sl_2)/(C=0) \xrightarrow{\sim} \mathcal{D}_{\mathbb{P}^1}^{glob}$.