Appendix to lecture 10: Hypergeometric function

Let $X = \mathbb{C}^2 \setminus 5$ lines: $\begin{matrix} z=0 \\ z=1 \\ t=0 \\ t=1 \\ t=z \end{matrix}$, $Y = \mathbb{C} \setminus \{0,1\}$.
$\underset{(z,t)}{\psi}$

Then we have a projection $\pi: X \to Y$, $\pi(z,t) = z$.
Let $a, b, c \in \mathbb{C}$ and $M_{a,b,c}$ be the right $\mathcal{D}_X$-module generated by the 2-form
$\omega_{a,b,c} = t^a (t-1)^b (t-z)^c \, dz dt$. This is a multivalued 2-form.

Let us compute $\pi_* M_{a,b,c}$, ~~for generic a,b,c~~. By applying the definition, we see (exercise) that $L^{-i}\pi_* M_{a,b,c}$ ~~$= H_{dR}^{1-i}$~~ is an $\mathcal{O}$-coherent $\mathcal{D}_Y$-module, whose fiber a $z \in \mathbb{C}\setminus\{0,1\}$ is $H^{1-i}_{dR}(M_{a,b,c,z})$,
where $M_{a,b,c,z}$ is the right $\mathcal{D}$-module on $\pi^{-1}(z) = \mathbb{C} \setminus \{0,1,z\}$ generated by the multivalued 1-form $t^a(t-1)^b(t-z)^c dt$.
The de Rham complex computing this cohomology is the complex

~~$0 \to \mathbb{C}[t^{\pm 1}, \frac{1}{t-1}]\Psi_{a,b,c} \xrightarrow{d} \mathbb{C}[t^{\pm 1}, \frac{1}{t-1}]\Psi_{a,b,c}\, dt \to 0$~~

~~with the usual differential d, where $\Psi_{a,b,c} = t^a(t-1)^b(t-z)^c$.~~

$$0 \to \mathbb{C}\left[t, \tfrac{1}{t}, \tfrac{1}{t-1}, \tfrac{1}{t-z}\right]\psi_{a,b,c} \xrightarrow{d} \mathbb{C}\left[t, \tfrac{1}{t}, \tfrac{1}{t-1}, \tfrac{1}{t-z}\right]\psi_{a,b,c}\, dt \to 0,$$

with the usual differential d, where

$\psi_{a,b,c} = t^a (t-1)^b (t-z)^c$ (so it depends on a,b,c only modulo $\mathbb{Z}$).

Since

$$d\psi_{a,b,c} = \left(\frac{a}{t} + \frac{b}{t-1} + \frac{c}{t-z}\right)\psi_{a,b,c}\, dt,$$

this complex can be rewritten as the usual de Rham complex with twisted differential:

$$0 \to \mathbb{C}\left[t, \tfrac{1}{t}, \tfrac{1}{t-1}, \tfrac{1}{t-z}\right] \xrightarrow{d_{a,b,c}} \mathbb{C}\left[t, \tfrac{1}{t}, \tfrac{1}{t-1}, \tfrac{1}{t-z}\right] dt \to 0$$

where $d_{a,b,c} = d + \left(\frac{a}{t} + \frac{b}{t-1} + \frac{c}{t-z}\right) dt$.

It is clear that H^0 of this complex is $\mathbb{C}$ if $a,b,c \in \mathbb{Z}$ and 0 otherwise (the only function in $\mathbb{C}\left[t, \tfrac{1}{t}, \tfrac{1}{t-1}, \tfrac{1}{t-z}\right]\psi_{a,b,c}$ killed by d up to scale can be $f=1$, and it belongs to this space iff $a,b,c \in \mathbb{Z}$). Also if $a,b,c = 0$, then H^1 of this complex equals $H^1(\mathbb{C} \setminus \{0,1,z\}) \cong \mathbb{C}^3$, with basis $\frac{dt}{t}$, $\frac{dt}{t-1}$, $\frac{dt}{t-z}$.

Indeed, these forms have independent

residues around $0, 1, z$, and any other form is a linear combination of these plus a form with zero residues, whose integral is a rational function.

Now, if a, b, c are not all in $\mathbb{Z}$, by Euler characteristic argument, $H^1 \cong \mathbb{C}^2$. It may be checked that a basis of H^1 is formed by

$\alpha_0 = \psi_{a,b,c} \frac{dt}{t}$ and $\alpha_1 = \psi_{a,b,c} \frac{dt}{t-1}$ for generic a, b, c.

However, $\psi_{a,b,c} \frac{dt}{t-z}$ is dependent on them, as

$$\psi_{a,b,c}\left(a\frac{dt}{t} + b\frac{dt}{t-1} + c\frac{dt}{t-z}\right) = d\psi_{a,b,c}.$$

Thus, we see that for generic a, b, c, $\pi_{*0} M_{a,b,c}$ $\left(= M_{a,b,c}/\partial_t M_{a,b,c} : \partial_t\right)$ is an $\mathcal{O}$-coherent $\mathcal{D}$-module on $\mathbb{C} \setminus \{0, 1\}$ of rank 2. Let us calculate this $\mathcal{D}$-module more explicitly.

For this purpose, we need to compute

$$\alpha_0 = \partial_z \psi_{a,b,c} \frac{dt}{t} = -c\,\psi_{a,b,c} \frac{1}{t-z} \cdot \frac{dt}{t}$$

$$\alpha_1 = \partial_z \psi_{a,b,c} \frac{dt}{t-1} = -c\,\psi_{a,b,c} \frac{1}{t-z} \frac{dt}{t-1},$$

and express them via α_0 and α_1.

We have

$$\partial_z \alpha_0 \cdot \cancel{dz} = \frac{c}{z}\psi_{a,b,c}\left(\frac{dt}{t-z} - \frac{dt}{t}\right) = \frac{c}{z}\left(+\frac{a}{c}\frac{dt}{t} + \frac{b}{c}\frac{dt}{t-1} + \frac{dt}{t}\right)$$

$$= \left(+\frac{a+c}{z}\frac{dt}{t} + \frac{b}{z}\frac{dt}{t-1}\right)\psi_{a,b,c} = +\frac{a+c}{z}\alpha_0 + \frac{b}{z}\alpha_1$$

$$\partial_z \alpha_1 \cdot \cancel{dz} = c\,\psi_{a,b,c}\left(\frac{dt}{t-z} - \frac{dt}{t-1}\right) = \cancel{\ldots}$$

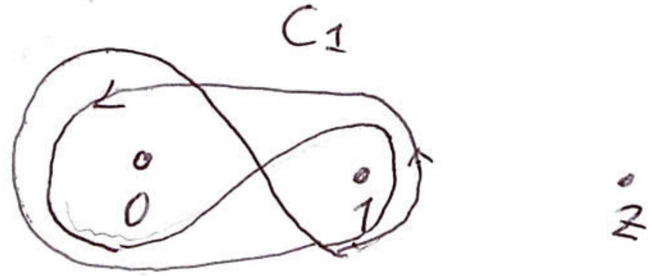
$$= +\frac{a}{z-1}\alpha_0 + \frac{b+c}{z-1}\alpha_1.$$

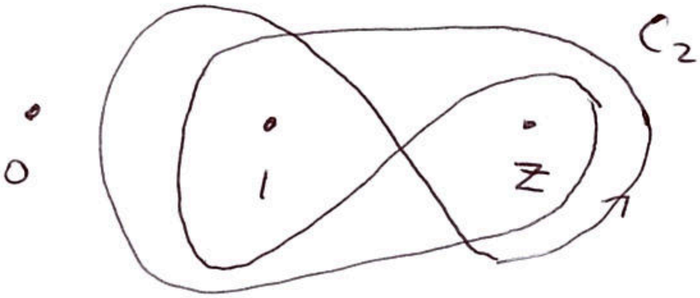
So the connection we get downstairs is

$$d - \begin{pmatrix} \frac{a+c}{z} & \frac{b}{z} \\ \frac{a}{z-1} & \frac{b+c}{z-1} \end{pmatrix}$$

This connection is called the hypergeometric connection. Up to conjugation by a scalar function, it's the most general connection $d + \frac{A}{z} + \frac{B}{z-1}$ (up to conjugation).

How to construct flat sections of this connection? For this purpose we need a cycle C on which we can integrate ~~functions~~ forms of the form $\psi_{a,b,c}\,\omega$, where ω is a rational form. Such a cycle can be taken to be a Pochhammer loop:



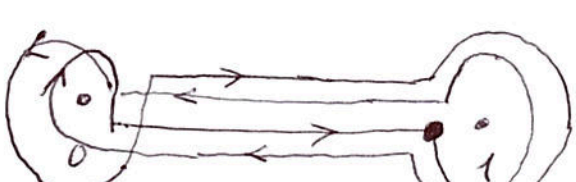


So flat sections are

$$F_1 = \begin{pmatrix} \int_{C_1} \alpha_0 \\ \int_{C_1} \alpha_1 \end{pmatrix}, \quad F_2 = \begin{pmatrix} \int_{C_2} \alpha_0 \\ \int_{C_2} \alpha_1 \end{pmatrix}$$

Note that if the real parts of a, b, c are large enough, we can contract C_1 to interval $[0,1]$, and up to a scalar we can replace $\int_{C_1}$ by $\int_0^1$:

$$\int_{C_1} \alpha_0 = \left(1 - e^{2\pi i b}\right)\left(1 - e^{2\pi i a}\right) \int_0^1 \alpha_0$$



The function $F = \int_0^1 t^a (t-1)^b (t-z)^c \frac{dt}{t}$ up to renormalization and reparametrization is called the Gauss hypergeometric function. It satisfies a 2nd order ODE which now can get by eliminating $\int_0^1 \alpha_1$, and $\mathcal{D}(\mathbb{C}\setminus\{0,1\})\cdot F$ is isomorphic to $\pi_{*0} M_{a,b,c}$. The $\mathcal{D}$-module $\pi_{*0} M_{a,b,c}$ is called the hypergeometric $\mathcal{D}$-module. (it's irreducible).

<u>General picture</u>: Suppose $\pi : X \twoheadrightarrow Y$ is a submersion of smooth varieties. So $\forall y \in Y$, $\pi^{-1}(y) = X_y$ is smooth. Suppose we have an $\mathcal{O}$-coherent $\mathcal{D}$-module on X, call it M. Then $\pi_{*0} M\big|_y = H^{top}_{dR}(X_y, M)$. This is an $\mathcal{O}$-coherent $\mathcal{D}$-module on Y, whose fibers are $H^{top}_{dR}(X_y, M)$.

If $\alpha_1, \dots, \alpha_N$ is a basis of this space (near some y), then if y_i are coordinates on y, we can compute

$$\partial_{y_i} \alpha_j = \sum a_{i,jl}(y)\, \alpha_l .$$

So we get a flat connection $d + \sum A_i \, dy_i$ which is called the Gauss-Manin connection. Flat sections of this connection correspond to elements $C_i \in H_{top}(X_y, M)$ as $\left(\int_{C_i} \alpha_1, \dots, \int_{C_i} \alpha_n\right)$.

Basic example: $X = \mathbb{C}^{n+m} \setminus$ hyperplanes H_j, $\pi : \mathbb{C}^{n+m} \to \mathbb{C}^n$ projection, $Y = \pi(X)$, so we have $\pi : X \to Y$. Let ℓ_j be the affine linear functions defining H_j, and $\lambda_j \in \mathbb{C}$. Let $M_\lambda = \langle \underbrace{\prod \ell_j^{\lambda_j}(z,t)\, dz_1 \cdots dz_n\, dt_1 \cdots dt_m}_{\Psi\, dz\, dt} \rangle$.

Then $\pi_{*0} M_\lambda$ admits a Gauss-Manin connection.

Ex. $$\Psi(z,t) = \prod_{i,j} (t_i - z_j)^{\mu_j} \prod_{\substack{i,j \\ i<j}} (t_i - t_j)^{\gamma}.$$

Then the corr. Gauss-Manin connection is the Knizhnik-Zamolodchikov connection.