Lecture 10.

Now we prove the theorem we stated above that direct and inverse images preserve holonomic modules.

Last time we defined two functors for a closed embedding $i: X \to Y$ of smooth varieties: $i^{*0}(M) = \mathcal{D}_{X\to Y} \underset{\mathcal{D}_Y}{\otimes} M = \mathcal{O}(X) \underset{\mathcal{O}(Y)}{\otimes} M$ and $i^{!0}(M) = Hom_{\mathcal{O}(Y)}(\mathcal{O}(X), M)$. The first functor is right exact, and the second one is left exact. What is the relation between these functors?

<u>Lemma 1</u> $i^{!0} = L^{\dim X - \dim Y} i^{*0}$
$i^{*0} = R^{-\dim X + \dim Y} i^{!0}$

(this is true both for $\mathcal{O}$-modules and for $\mathcal{D}$-modules)

<u>Proof</u>. We can construct the isomorphism locally (and then show it agrees on intersections), As in the proof of Kashiwara's thm, we can assume that $X \subset Y$ has codimension 1 and is defined by the equation $f=0$. By def. we have a short exact sequence

$$0 \to \mathcal{O}(Y) \xrightarrow{f} \mathcal{O}(Y) \to \mathcal{O}(X) \to 0.$$

Tensoring with M over $\mathcal{O}(Y)$, we get

$$M \xrightarrow{f} M \longrightarrow \underbrace{\mathcal{O}(X) \underset{\mathcal{O}(Y)}{\otimes} M}_{= i^{*0}M} \longrightarrow 0.$$

So by definition $L^{-1} i^{*\circ} M = \operatorname{Ker} f = i^{!\circ} M$, and $i^{*\circ} M = \operatorname{Coker} f = R^1 i^{!\circ} M$ (as by taking Hom into M we also have

$$M \xleftarrow{f} M \leftarrow \operatorname{Hom}_{\mathcal{O}(Y)}(\mathcal{O}(X), M) \leftarrow 0).$$

This proves the lemma.

<u>Lemma 2</u>. If $i: X \hookrightarrow Y$ is a closed embedding of smooth varieties, then the functors $i^{!\circ}$, $i_{*\circ}$ maps holonomic $\mathcal{D}$-modules on X to holonomic $\mathcal{D}$-modules on Y supported on X, and vice versa.

<u>Proof</u>. As before, we may assume that X has codimension 1 in Y and is given by the equation $f=0$. In the proof of Kashiwara's theorem, we showed that $i_{*\circ} M = \bigoplus_{j \geq 0} M \partial^j$, where ∂ is a vector field such that $\partial(f) = 1$. Note that $\forall x \in X$, $T_x^* Y = T_x^* X \oplus k \cdot df_x$. It's clear that $SS(i_{*\circ} M) = SS(M) \times k \cdot \cancel{df}$, so $d(i_{*\circ} M) = d(M) + 1$. Similarly, by Kashiwara's theorem, if $N \in \mathcal{M}_X^2(\mathcal{D}_Y)$, then $SS(i^{!\circ} N) = SS(N) / k \cdot df$, so $d(i^{!\circ} N) = d(N) - 1$. This proves Lemma 2. $\blacksquare$

<u>Lemma 3.</u> The functor $i^{!0}: \mathcal{M}^z(D_Y) \to \mathcal{M}^z(D_X)$ for a closed embedding $i: X \hookrightarrow Y$ of smooth varieties maps holonomic modules to holonomic ones.

<u>Proof.</u> For $N \in \mathcal{M}^z(D_Y)$, let N' be the set of vectors killed by some power of the ideal I_X of $X \subset Y$. Then $N' \subset N$ is a submodule, so it is holonomic, and $i^{!0}N = i^{!0}N'$. But N' is supported on X, so the statement follows from Lemma 2.

<u>Lemma 4.</u> Let $i: \overset{X}{\overset{\|}{\mathbb{A}^m}} \hookrightarrow \overset{Y}{\overset{\|}{\mathbb{A}^n}}$ be an affine linear inclusion. Then $i^{*0}: \mathcal{M}^z(D_Y) \to$ $\to \mathcal{M}^z(D_X)$ maps holonomic modules to holonomic ones.

<u>Proof.</u> It suffices to assume that $m = n-1$. Let $x_1, \ldots, x_n$ be coordinates on $\mathbb{A}^n$, and $\mathbb{A}^{n-1}$ be defined by $x_n = 0$. Then $i^{*0}N = N/x_n N$.
Let N be holonomic on $\mathbb{A}^n$.
Let F be a good filtration on N.

We have $\dim F_j N = c(N)\frac{j^n}{n!} + \cdots$ (as N is holonomic). Then

$$\dim F_j\left(N/x_n N\right) \leq \dim F_j \frac{N}{x_n F_{j-1} N} = \dim F_j N - \dim x_n F_{j-1} N$$

(as we have an epimorphism $F_j N/x_n F_{j-1} N \to F_j\left(N/x_n N\right)$).
Now, since i^{*0} is right exact, we may assume that N is irreducible. If N is concentrated on X, $i^{*0}N = 0$, so we may assume N is not concentrated on X. Then $x_n : N \to N$ is injective, so

$$\dim F_j\left(N/x_n N\right) \leq \dim F_j N - \dim F_{j-1} N, \text{ i.e.}$$

$$\dim F_j\left(N/x_n N\right) \leq c(N)\frac{j^{n-1}}{(n-1)!} + \cdots$$

By what we showed before, this implies that $M/x_n M = i^{*0} N$ is holonomic on $\mathbb{A}^{n-1}$ (even though the filtration F_j may not be good). ∎

<u>Lemma 5</u>. In Lemma 4, all the derived functors of i^{*0} map holonomic modules to holonomic ones.

<u>Proof</u>. If $m = n-1$, by lemma 1, $L^{-1} i^{*0} = i^{!0}$, so

the statement follows for $m=n-1$ by lemmas 3, 4. For $m<n-1$, the statement follows by induction in $n-m$, using the spectral sequence of composition of functors. ⊠

<u>Fourier transform.</u> Let V be a f.d. vector space over k. Let $\mathcal{F}: \mathcal{D}_V \to \mathcal{D}_{V^*}$ be the automorphism such that

$$\mathcal{F}(f) = D_f, \quad \mathcal{F}(D_v) = -v, \quad f \in V^*, v \in V.$$

Then $\mathcal{F}$ defines a functor, also called $\mathcal{F}: \mathcal{M}(\mathcal{D}_V) \to \mathcal{M}(\mathcal{D}_{V^*})$ (by pullback).

<u>Lemma 6.</u> Let $\pi: V \to W$ be a linear map, and $\mathcal{F}_V, \mathcal{F}_W$ be Fourier transforms for V, W. ~~Then~~ Let $\pi^\vee: W^* \to V^*$ be the dual map. Then

$$\mathcal{F}_W \circ \pi_{*0} = \pi^{\vee *0} \circ \mathcal{F}_V$$

$$\mathcal{F}_V \circ \pi^{*0} = \pi^\vee_{*0} \circ \mathcal{F}_W.$$

<u>Proof.</u> It suffices to prove just the first equality. ~~We have~~ Any π is a composition ~~of~~ of a surjection and an injection, so it suffices to deal with each of these cases.

<u>Case 1.</u> $\pi: V \to W$ is surjective.

We may assume $\pi : \overset{V}{\overset{\shortparallel}{\mathbb{A}^n}} \to \overset{W}{\overset{\shortparallel}{\mathbb{A}^{n-1}}}$,
$\pi(x_1, \ldots, x_n) = (x_1, \ldots, x_{n-1})$.
Then $\mathcal{F}_W \circ \pi_{*0}(M) = \mathcal{F}_W\left(M/\partial_n M\right) = \mathcal{F}_V(M)/x_n \mathcal{F}_V(M)$
$= \pi^{\vee *0} \circ \mathcal{F}_V(M)$.

<u>Case 2.</u> $\pi : V \to W$ is injective. We may assume
$\pi : \mathbb{A}^{n-1} \to \mathbb{A}^n$, $\pi(x_1, \ldots, x_{n-1}) = (x_1, \ldots, x_{n-1}, 0)$.
Then
$$\mathcal{F}_W \circ \pi_{*0}(M) = \mathcal{F}_W\left(M \boxtimes \delta_0\right) = \mathcal{F}_V(M) \boxtimes \mathcal{O}_{\mathbb{A}^1}$$
$= \pi^{\vee *0} \mathcal{F}_V(M)$ ∎

<u>Lemma 7.</u> Let $\pi : \mathbb{A}^m \to \mathbb{A}^n$ be an affine map. Then the functors π_{*0}, π^{*0}, and all their derived functors map holonomic modules to holonomic ones.

<u>Proof.</u> Any affine map is a composition of an affine embedding and an affine projection. For embeddings the statement is known (Lemma 2, Lemmas 4,5). For projections, it follows from the embedding case by Lemma 6 (since Fourier transform clearly preserves holonomicity, as it rotates arithmetic singular support). ∎

Open embeddings. Let $j: U \hookrightarrow X$ be an embedding of an affine open set into a smooth variety.

~~Lemma 8.~~ The functor $j_{*0}: \mathcal{M}^r(\mathcal{D}_U) \to \mathcal{M}^r(\mathcal{D}_X)$ is given by $j_{*0}(M) = M \underset{\mathcal{D}_U}{\otimes} \mathcal{D}_{U \to X} =$

$$= M \underset{\mathcal{D}_U}{\otimes} \mathcal{O}(U) \underset{\mathcal{O}(X)}{\otimes} \mathcal{D}_X = M \underset{\mathcal{D}_U}{\otimes} \mathcal{D}_U = M.$$

So j_{*0} is just restriction of M to the subalgebra $\mathcal{D}_X \subset \mathcal{D}_U$.

Lemma 8. Let $j: U \hookrightarrow \mathbb{A}^n$ be an affine open embedding. Then j_{*0} (which is exact, as shown above) maps holonomic modules to holonomic ones.

Proof. ~~We~~ Let $U = \{x \in \mathbb{A}^n \mid f(x) \neq 0\}$, where f is a polynomial. Then consider $i: U \to \mathbb{A}^{n+1}$ given by $i(x) = \left(j(x), \frac{1}{f(x)}\right)$. This is a closed embedding, and $j = \pi \circ i$, where $\pi: \mathbb{A}^{n+1} \to \mathbb{A}^n$ is a projection. So $j_{*0} = \pi_{*0} \circ i_{*0}$, and the statement follows from Lemma 7 and Lemma 2. ■

Lemma 9. Let $i : X \hookrightarrow Y$ be a closed embedding of smooth varieties. Then the functor i^{*0} and all its derived functors map holonomic $\mathcal{D}$-modules to holonomic ones.

Proof. This is a local statement, so we may assume that $X \subset Y$ is codimension 1 and given by equation $f = 0$. Moreover we may assume that we have a coordinate system $x_1, .., x_n$ on Y such that $f = x_n$. Let $U \subset \mathbb{A}^n$ be the image of the map $(x_1, .., x_n) = \psi : Y \to \mathbb{A}^n$. Then $(x_1, .., x_n) = \varphi : Y \to U$ is a covering. So given a holonomic $\mathcal{D}_Y$-module N, $\varphi_{*0} N$ (the restriction of N to the subalgebra $\mathcal{D}_U$) is clearly holonomic (It's clear that $SS(\varphi_{*0} N) = \varphi(SS(N))$. So $\psi_{*0} N \in \mathcal{M}(\mathcal{D}_{\mathbb{A}^n})$ is also holonomic (by Lemma 8), as $\psi = j \circ \varphi$, $j : U \to \mathbb{A}^n$. Now, $i^{*0} N = N/fN = N/x_n N$, so ~~$\psi$~~ $N/x_n N$ is holonomic as a $\mathcal{D}_{\mathbb{A}^{n-1}}$-module by Lemma 4. Thus its singular support S has dimension $n-1$.

But we have a diagram

so $SS(i^{*0}N)$

$= \varphi^{-1}(S \cap T^*U) \subset T^*X.$

$$\begin{tikzcd}
 & X \arrow[dl, "\varphi"'] \arrow[d, "\psi"] \arrow[rr, hook] & & Y \arrow[dl, "\varphi"'] \arrow[d, "\psi"] \\
U \cap \mathbb{A}^{n-1} \arrow[r, hook] \arrow[rr, bend right] & \mathbb{A}^{n-1} \arrow[rr, bend right] & U \arrow[r, hook] & \mathbb{A}^n
\end{tikzcd}$$

Thus $\dim SS(i^{*0}N) = n-1$, so $i^{*0}N$ is holonomic. The same argument applies to derived functors. ■

<u>Lemma 10</u>. Let $\pi : X \times \mathbb{A}^n \to X$ be the projection. Then π_{*0} and its derived functors map holonomic modules to holonomic ones.

<u>Proof</u>. It's enough to assume that $n=1$. Let $\mathcal{F} : \mathcal{M}(\mathcal{D}_{X \times \mathbb{A}^1}) \to \mathcal{M}(\mathcal{D}_{X \times \mathbb{A}^1})$ be the functor of fiberwise Fourier transform ($\mathcal{F}(x) = \partial$, $\mathcal{F}(\partial) = -x$). Then it's shown as in L.6 that $\pi_{*0} = \pi^{\vee *0} \circ \mathcal{F}$, where $\pi^\vee : X \to X \times \mathbb{A}^1$ is the embedding. Thus, the statement follows from Lemma 9.

<u>Theorem 11</u>. Let $\pi : X \to Y$ be any morphism of affine varieties. Then the functors π_{*0} and π^{*0} and their derived functors map holonomic $\mathcal{D}$-modules to holonomic ones.

Proof. We can write π as a composition $\pi = p \circ (\pi, i)$, where $i : X \to \mathbb{A}^n$ is a closed embedding and $p : Y \times \mathbb{A}^n \to Y$ the projection. So it suffices to prove the theorem for closed embeddings and projections.

For closed embeddings and π_{*0} it is lemma 2. For closed embeddings and π^{*0} it is lemma 9. For projections it is lemma 10. We are done.

Corollary 12. If M is holonomic on X then $\dim H^i_{dR}(M) < \infty$. In particular for smooth X, $\dim H^i_{dR}(X) < \infty$ (algebraic De Rham Cohomology).

Remark. In fact, Grothendieck showed that $H^i_{dR}(X) \cong H^i(X, \mathbb{C})$ for complex varieties, but this uses resolution of singularities. Finite dimensionality is simpler, but still nontrivial.

<u>Remark</u>. Both Kashiwara's theorem and the theorem on preservation of holonomicity fail in characteristic p, for crystalline differential operators. E.g. if $i: pt \hookrightarrow \mathbb{A}^1$ is the embedding then $i_{*0}\, k = \delta_0$, but $i^{!0}\delta_0 = \{v \in \delta_0 \mid xv = 0\} = \{\delta^{(rp)}, r \geq 0\}$. So $i^{!0}$ and i_{*0} are not inverse to each other. Also δ_0 is finitely generated but $i^{!0}\delta_0$ is not. (in char 0 the Kashiwara equivalence maps finitely generated $\mathcal{D}$-modules to finitely generated ones, since f.g. $\Leftrightarrow$ Noetherian (satisfying ACC)) Also for $\pi: \mathbb{A}^1 \to pt$, $\pi_{*0}\Omega^1 = \Omega^1_{\mathbb{A}^1}{}^{\partial} = k[x^p]dx/x$, which is ∞-dimensional.

<u>Application</u>: Lie algebra coinvariants.

Let $\mathfrak{g}$ be a Lie algebra over k (char $k = 0$) and X an irred. affine variety over k. Suppose $\mathfrak{g}$ acts on X, i.e. we have a homomorphism $\rho: \mathfrak{g} \to \mathrm{Vect}(X)$. We say that the action is transitive if the map $\rho_x: \mathfrak{g} \to T_x X$ is surjective for all $x \in X$. It's easy to see that if the action is transitive then X is smooth, since $\dim \mathrm{Im}\, \rho_x$ is ~~lower~~ upper semicontinuous, while $\dim T_x X$ is lower semicontinuous.

<u>Ex.</u> $\mathrm{Vect}(X)$ acts transitively on X iff X is smooth

Indeed, one can show that any vector field on X is tangent to the singularity locus $\mathrm{Sing}(X)$. (this is false in char p).

<u>Def</u>. A $\mathfrak{g}$-orbit of X is a locally closed subvariety $Z \subset X$ such that $\rho_x(\mathfrak{g}) = T_x Z$ $\forall x \in Z$ (i.e. $\mathfrak{g}$ acts transitively on Z). I.e. Z is $\mathfrak{g}$-invariant and $\mathfrak{g}$ acts transitively on Z.

(E.-Schedler)

<u>Theorem 6</u>. If ρ has finitely many $\mathfrak{g}$-orbits i.e. X is a union of finitely many $\mathfrak{g}$-orbits, then the space $H_0(\mathfrak{g}, \mathcal{O}(X)) = \mathcal{O}(X)/\mathfrak{g}\cdot\mathcal{O}(X)$ is finite dimensional.

<u>Proof</u>. Let ρ be any $\mathfrak{g}$-action on X. Consider the $\mathcal{D}$-module M_ρ on X defined by $M_\rho = \tilde{\mathcal{D}}_X/\mathfrak{g}\cdot\tilde{\mathcal{D}}_X$ (recall that $\tilde{\mathcal{D}}_X$ is the $\mathcal{D}$-module on X representing the functor of global sections, which has an action of Grothendieck differential operators, in particular $\mathrm{Vect}(X)$ and $\mathfrak{g}$). Explicitly, if $i: X \hookrightarrow V$ is a closed embedding of X into a vector space V, then lift $\rho: \mathfrak{g} \to \mathrm{Vect}(X)$ to a linear map $\tilde{\rho}: \mathfrak{g} \to \mathrm{Vect}_{X}(V)$ vector fields on V tangent to X (i.e. preserving the ideal I_X of X).

Then $M_\rho = \mathcal{D}_V / I_X \mathcal{D}_V + \tilde{\rho}(\mathfrak{g}) \mathcal{D}_V$

(as $\tilde{\mathcal{D}}_X = \mathcal{D}_V / I_X \mathcal{D}_V$).

<u>Proposition 7.</u> Suppose X is a union of finitely many $\mathfrak{g}$-orbits. Then $SS(M_\rho)$ is contained in the union of conormal bundles of the $\mathfrak{g}$-orbits.

<u>Proof.</u> $\mathrm{gr}\, M_\rho$ is a quotient of

$$\mathcal{O}(T^*V) / I_X \mathcal{O}(T_*V) + \tilde{\rho}(\mathfrak{g}) \cdot \mathcal{O}(T_*^*V) = R.$$

Closed points of $\mathrm{Spec}\, R$ are points $(x, p) \in V \times V^*$ such that $x \in X$ and $\langle p, \rho_x(a) \rangle = 0$ $\forall a \in \mathfrak{g}$, so they belong to the union conormal bundles of Z_i, the $\mathfrak{g}$-orbits of ρ.

<u>Corollary 8.</u> If X consists of finitely many $\mathfrak{g}$-orbits then M_ρ is holonomic.

<u>Proof.</u> Conormal bdles are Lagrangian, so SS is Lagrangian.

Now we prove theorem 6. We have

$$\mathcal{O}(X) / \mathfrak{g} \cdot \mathcal{O}(X) = \mathcal{O}(V) / I_X \mathcal{O}(V) + \tilde{\rho}(\mathfrak{g}) \mathcal{O}(V)$$

$= M_\rho \otimes_{\mathcal{D}_V} \mathcal{O}(V) = \pi_{*0} M_\rho$, where $\pi : V \to pt$.

since M_ρ is holonomic, by thm on preservation of holonomicity, $\pi_{*0} M_\rho$ is finite dimensional, so $\mathcal{O}_X / \mathfrak{g} \mathcal{O}_X$ is finite dimensional. $\blacksquare$

Example. Let X be a Poisson variety. In this case, take $\mathfrak{g} = \mathcal{O}_X$, with $\rho: \mathfrak{g} \to \mathrm{Vect}\, X$ be defined by $\rho(f) \cdot g = \{f, g\}$. In this case, $\mathfrak{g}$-orbits are called symplectic leaves.

Corollary 9. If X has finitely many symplectic leaves then $\mathcal{O}_X / \{\mathcal{O}_X, \mathcal{O}_X\}$ is finite dimensional. (Alev–Farkas conjecture).

Corollary 10. Suppose a finite group G acts on a symplectic variety Y (e.g. a vector space). Then $\mathcal{O}(Y)^G / \{\mathcal{O}(Y)^G, \mathcal{O}(Y)^G\}$ is finite dimensional.

Proof. Y/G is a Poisson variety, with finitely many symplectic leaves (namely, one can show that symplectic leaves are labeled by conjugacy classes of subgroups $H \subset G$ such that $H = \mathrm{Stab}(v) \subset G$ for some $v \in V$); more precisely, $Z_H = \{v \in V, \mathrm{Stab}(v) \text{ is conjugate to } H\}/G$. $\blacksquare$