Algebraic D-modules

Lecture 1.

One of the motivations for development of the theory of D-modules (i.e., modules over algebras of differential operators) was the following analytic problem, raised by Gelfand and Sato.

Let \( p: \mathbb{R}^n \rightarrow \mathbb{R} \) be a polynomial, and \( U \) be a connected component of the open set \( p \neq 0 \). Let \( f \in C_c^\infty(\mathbb{R}^n) \) be a smooth function with compact support. Consider the integral

\[
I(\lambda) = \int_U |p(x)|^\lambda f(x) \, dx
\]

as a function of the complex variable \( \lambda \).

Clearly, \( I(\lambda) \) is holomorphic if \( \Re(\lambda) > 0 \) (in fact, in a slightly wider strip). Consider the following question:

**Question 1:** Does \( I(\lambda) \) admit a meromorphic continuation to the entire complex plane? In other words, is the distribution \( |p(x)|^\lambda \cdot f(x) \) meromorphic in \( \lambda \)?

**Example.** \( n = 1, \ p(x) = x, \ u = \mathbb{R}_+ \).

Then \( \int_0^\infty x^\lambda f(x) \, dx \). Note that \( I(\lambda) \) is analytic for \( \Re(\lambda) > -1 \).

In this case we can prove in an elementary way

**Proposition 1:** If \( I(\lambda) \) is meromorphic on \( \mathbb{C} \)
with possible simple poles at \( \lambda = -1, -3, -5 \ldots \) and no other singularities.

**Proof.** Let \( \text{Re} \lambda > 0 \). Then integrating by parts, we get

\[
I_f(\lambda) = \int_0^\infty x^\lambda f(x) \, dx = \frac{1}{\lambda+1} \int_0^\infty \frac{d}{dx} (x^{\lambda+1}) \cdot f(x) \, dx
\]

\[
= -\frac{1}{\lambda+1} \int_0^\infty x^{\lambda+1} f'(x) \, dx = -\frac{1}{\lambda+1} I_f'(\lambda+1).
\]

This implies the statement (indeed, we get \( I_f(\lambda) = (-1)^N \frac{1}{(\lambda+1) \cdots (\lambda+N)} I_f(\lambda+N) \), and \( I_f(\lambda+N) \) is holomorphic for \( \text{Re} \lambda > -N-1 \)).

**Remark.** In fact this also works if \( f \) is rapidly decaying along with all its derivatives. E.g. we can take \( f(x) = e^{-x} \) (more precisely, some extension of this function to a \( C^\infty \)-function on \( \mathbb{IR} \) rapidly decaying with derivatives). Then \( I_f(\lambda) = \int_0^\infty x^\lambda e^{-x} \, dx = \Gamma(\lambda+1) \), and Prop 1 implies that \( \Gamma(\lambda) \) is meromorphic with simple poles at \( 0, -1, -2, \ldots \).

This essentially says that the answer to Question 1 is "yes" in the 1-dimensional case. But in higher dimensions the problem is more difficult. Nevertheless, we have the following theorem.
Theorem 2. If $(\lambda)$ is meromorphic in $\mathbb{C}$ with poles on a finite number of one-sided arithmetic progressions with step 1 going in the negative direction. This theorem is due to Atiyah and Bernstein, and the first proofs were based on Hironaka resolution of singularities, which is a very difficult theorem. But there is a purely algebraic proof of this theorem, due to Bernstein, which is much easier and is based on the theory of $D$-modules.

We will explain this proof here. First we will formulate an algebraic statement which implies Theorem 2.

Let $D$ be the algebra of differential operators on $\mathbb{R}^n$ with polynomial coefficients, i.e. the algebra generated by $x_i$ and $\frac{\partial}{\partial x_i}$, $i=1,\ldots,n$.

Theorem 3. There exists $L \in D[\lambda]$ and $b(\lambda) \in \mathbb{C}[\lambda]$ such that

$$L (p^{\lambda+1}) = b(\lambda) p^\lambda.$$ 

(E.g. for $p(x) = x$, $n=1$, $L = \frac{d}{dx}$ and $b(\lambda) = \lambda + 1$.)

We claim that Theorem 3, which is purely algebraic, implies Theorem 2.
Indeed, we have
\[
I_f(x) = \int_{-\infty}^{\infty} \left| p(x) \right|^2 f(x) \, dx =
\]
\[
= \pm \frac{1}{b(x)} \int_{-\infty}^{\infty} \left| \lambda p(x) \right|^{\lambda+1} \cdot f(x) \, dx
\]
\[
= \pm \frac{1}{b(x)} \int_{-\infty}^{\infty} \left| p(x) \right|^{\lambda+1} \cdot L^* f(x) \, dx,
\]
and the proof proceeds as in the 1-dimensional case; the arithmetic progressions are \( a, x_k \)...

We will now reformulate Theorem 3 in terms of modules over the algebra \( D \).

More precisely, let \( \mathcal{D} = D \otimes C(\lambda) \), algebra over the field \( C(\lambda) \). Let \( M_p \) be the \( \mathcal{D} \)-module spanned by formal expressions
\[
q \cdot p^{\lambda-i}, \quad i \in \mathbb{Z}, \quad q \in \mathbb{C}[x_1, \ldots, x_n],
\]
with relations
\[
q \cdot p^{\lambda-i+1} = q \cdot p^{\lambda-i} \quad (i.e. \quad M_p = C[x_1, \ldots, x_n][\frac{1}{p}] \cdot p^\lambda)
\]

They is indeed a \( \mathcal{D} \)-module, with the action of \( \mathcal{D} \) defined in a natural way.

**Theorem 4.** \( M_p \) is a finitely generated \( \mathcal{D} \)-module.

Let us show that Theorem 4 implies Theorem 3 (in fact, they are equivalent).
let $M_i$ be the submodule of $M_{r_i}$
generated by $p^{r_i-j}$. Then $M_i < M_{r_i+1}$,
and $M_{r_i} = \bigcup M_i$. Assume that $M_{r_i}$
is finitely generated. This implies that
there is $j$ such that $M(r_i) = M_j$.
In other words, $M(r_i)$ is generated by
$p^{r_i-j}$, and thus there is $L \in \mathbb{Z}$ such that
$p^{r_i-j-1} = \sum p^{r_i-j}$
Let $\sigma : \mathbb{C}(x) \to \mathbb{C}(\lambda)$, $\sigma(x) = \lambda + j + 1$
Let $\sigma(L) = \frac{L}{b(\lambda)}$, where $L \in \mathbb{D}[\lambda]$.
Then $\frac{L}{b(\lambda)} p^{r_i-j} = p^{r_i-j}$, so $L p^{r_i-j+1} = b(\lambda) p^{r_i}$, and
Theorem 3 follows.
We now want to prove Theorem 4.
For this we’ll need some machinery.
If $\mathbb{A}$ is an associative algebra, by
an increasing filtration we will mean a collection of subspaces $F_i : V \subseteq V$, i\geq 0
such that
1) $F_0 V \subseteq F_i V \subseteq \ldots$
2) $U_i F_i V = V$
We will agree that $F_{i+1} V = 0$ and $F_i V = 0$ for $i > 0$. 

If $V = A$ is an associative algebra, we'll require that $1 \in F_0 A$ and $F_i A \cdot F_j A \subseteq F_{i+j} A$.

Associated graded space:

$$gr^F V = \bigoplus_{i=0}^\infty F_{i+1} V / F_i V.$$  \hspace{1cm} gr^F_i V.

If $V = A$ is an algebra, then so is $gr^F A$ (associated graded algebra). We have $gr^F_i A \cdot gr^F_j A \subseteq gr^F_{i+j} A$.

If $M$ is a left module over a filtered algebra $A$, then an increasing filtration of $M$ is a filtration on $M$ as a vector space, such that $F_i A \cdot F_j M \subseteq F_{i+j} M$.

Then $gr^F M$ is a graded $gr^F A$-module:

$$gr^F_i M \cdot gr^F_j M \subseteq gr^F_{i+j} M.$$  \hspace{1cm} gr^F_i j M \subseteq gr^F_{i+j} M.$$

**Def.** (1) A filtration $F$ on $M$ is called **good** if $gr^F M$ is a finitely generated $gr^F A$-module.

(2) Filtrations $F$ and $F'$ on $M$ are called **equivalent** if there exist $i_0, j, j_1 \in \mathbb{Z}_+$ such that $F_{i-j_0} M \subseteq F_j M \subseteq F_{i+j_1} M$.
Proposition 5. Let $F$ be a good filtration on $M$, and $\overline{m_1}, \ldots, \overline{m_n}$ be generators of $gr^F M$, over $gr^A$ of degrees $d_1, \ldots, d_n$. Let $m_1, \ldots, m_n$ be any lifts of $\overline{m_1}, \ldots, \overline{m_n}$.
Then for any $j \geq 0$,

$$F_j M = F_{j-d_1} M_1 + \cdots + F_{j-d_n} M_n.$$  

In particular, $M$ is a finitely generated $A$-module with generators $m_1, \ldots, m_n$.

(2) If $F'$ is another filtration on $M$ then $3k$ such that $\forall j$ $F_j M \leq F_{j+k} M$.

Proof. (1) We have $gr^F F_j M = gr^F F_{j-d_1} M_1 + \cdots + gr^F F_{j-d_n} M_n$ so the statement follows by induction on $j$.

(2) let $k$ be such that $m_i \in F_{j+k} M \forall i$. Then by (1) $F_j M \leq F_{j+k} M \forall i$.

Corollary 6. If $F, F'$ be good filtrations on $M$ then they are equivalent.

Main example: $D$ is the algebra of polynomial differential operators in $n$ variables. There are two important filtrations on $D$:

1) Bernstein filtration: $F_i D$ is the span of all monomials in degree $i$ in $x_i$ and $D_i$, i.e. image of $F_i D \otimes \mathbb{R}$ under multi-
2) Geometric filtration (filtration by order of differential operator)

\[ D_0 \subset D_1 \subset \ldots \quad D = \bigcup_{i \geq 0} D_i. \]

\( D_i = \) span of all monomials \( x_{i_1} \ldots x_{i_k} \) containing \( \leq i \) derivatives.

**Lemma 7 (easy).** For any \( L \in D \) there is a unique representation

\[ L = \sum_{1 \leq i_1 \leq \ldots \leq i_k \leq n} P_{i_1, \ldots, i_k} (x_1, \ldots, x_n) \partial_{i_1} \ldots \partial_{i_k}. \]

**Corollary 8.** For both filtrations, \( \text{gr} \ D = \mathbb{C}[x_1, \ldots, x_n, \xi_1, \ldots, \xi_n] \), where \( \xi_i \) are the images of \( \partial_i \).

**Proof.** We need to show that \( [\xi_j, x_i] = 0 \).

This is clear, since \( [\partial_j, x_i] = \delta_{ij} \), element of lower degree.

**Remark.** In fact, for Bernstein filtration we get a little more: \( [F_i \partial_j, x_i] \leq F_{i+j} - 2D. \)

**Prop 9.** Any f.g. \( A \)-module \( M \) has a good filtration.

**Proof.** Let \( m_1, \ldots, m_n \) be generators of \( M \), and let \( F_i M = F_i A \cdot m_1 + \ldots + F_i A \cdot m_n \). Then the images \( \overline{m}_i \) of \( m_i \) in \( \text{gr}_i M \) are generators of \( \text{gr} M \) over \( \text{gr} A. \)

**Ex.** \( A = \mathbb{C}[t], \ M = A, \ F_i M = F_i M = \ldots = M. \) Then \( \text{gr} M = M, \) and \( t/1 M = 0. \) So this is not a good filtration.