Algebraic D-modules.

Lecture 1.

One of the motivations for development of the theory of D-modules (i.e. modules over algebras of differential operators) was the following analytic problem, raised by Gelfand and Sato.

Let $p: \mathbb{R}^n \to \mathbb{R}$ be a polynomial, and U be a connected component of the open set $p \neq 0$. Let $f \in C_c^\infty(\mathbb{R}^n)$ be a smooth function with compact support. Consider the integral

$$I_f(\lambda) = \int_U |p(x)|^\lambda f(x)\,dx$$

as a function of the complex variable λ.

Clearly, $I(\lambda)$ is holomorphic if $\mathrm{Re}(\lambda) > 0$ (in fact, in a slightly wider strip.

Question 1: Does $I_f(\lambda)$ admit a meromorphic continuation to the entire complex plane? In other words, is the distribution $|p(x)|^\lambda \cdot \chi_U$ meromorphic in λ?

Example. $n = 1$, $p(x) = x$, $U = \mathbb{R}_+$.

Then $I_f(\lambda) = \int_0^\infty x^\lambda f(x)\,dx$. Note that $I_f(\lambda)$ is analytic for $\mathrm{Re}\,\lambda > -1$.

In this case we can prove in an elementary way

Proposition 1. $I_f(\lambda)$ is meromorphic on $\mathbb{C}$

with possible simple poles at $\lambda = -1, -2, -3, \ldots$ and no other singularities.

<u>Proof.</u> Let $\mathrm{Re}\,\lambda > 0$. Then integrating by parts, we get

$$I_f(\lambda) = \int_0^\infty x^\lambda f(x)\,dx = \frac{1}{\lambda+1}\int_0^\infty \frac{d}{dx}(x^{\lambda+1})\cdot f(x)\,dx$$

$$= -\frac{1}{\lambda+1}\int_0^\infty x^{\lambda+1} f'(x)\,dx = -\frac{1}{\lambda+1} I_{f'}(\lambda+1).$$

This implies the statement (indeed, we get $I_f(\lambda) = (-1)^N \frac{1}{(\lambda+1)\cdots(\lambda+N)} I_{f^{(N)}}(\lambda+N)$, and $I_{f^{(N)}}(\lambda+N)$ is holomorphic for $\mathrm{Re}\,\lambda > -N-1$).

<u>Remark.</u> In fact this also works if f is rapidly decaying along with all its derivatives. E.g. we can take $f(x) = e^{-x}$ (more precisely, some extension of this function to a C^∞-function on $\mathbb{R}$ rapidly decaying with derivatives). Then $I_f(\lambda) = \int_0^\infty x^\lambda e^{-x}\,dx = \Gamma(\lambda+1)$, and Prop 1 implies that $\Gamma(\lambda)$ is meromorphic with simple poles at $0, -1, -2, \ldots$

This essentially says that the answer to Question 1 is "yes" in the 1-dimensional case. But in higher dimensions the problem is more difficult. Nevertheless, we have the following theorem.

<u>Theorem 2.</u> For any n $I_f(\lambda)$ is meromorphic in $\mathbb{C}$ with poles on a finite number of one-sided arithmetic progressions with step 1 going in the negative direction.

This theorem is due to Atiyah and Bernstein, and the first proofs were based on Hironaka resolution of singularities, which is a very difficult theorem. But there is a purely algebraic proof of this theorem, due to Bernstein, which is much easier and is based on the theory of D-modules. We will explain this proof here. First we will formulate an algebraic statement which implies Theorem 2.

Let $\mathcal{D}$ be the algebra of differential operators on $\mathbb{R}^n$ with polynomial coefficients, i.e. the algebra generated by x_i and $\frac{\partial}{\partial x_i}$, $i=1,\dots,n$.

<u>Theorem 3.</u> There exists $L \in \mathcal{D}[\lambda]$ and $b \in \mathbb{C}[\lambda]$ such that

$$L(p^{\lambda+1}) = b(\lambda)\, p^{\lambda}.$$

(E.g. for $p(x)=x$, $n=1$, $L=\frac{d}{dx}$ and $b(\lambda)=\lambda+1$).

We claim that Theorem 3, which is purely algebraic, implies Theorem 2

Indeed, we have

$$I_f(\lambda) = \int_U |p(x)|^\lambda f(x)\,dx =$$

$$= \pm\frac{1}{b(\lambda)} \int_U L\,|p(x)|^{\lambda+1} \cdot f(x)\,dx$$

$$= \pm\frac{1}{b(\lambda)} \int_U |p(x)|^{\lambda+1} \cdot L^* f(x) \cdot dx\,,$$

and the proof proceeds as in the 1-dimensional case; the arithmetic progressions are $\alpha, \alpha-1, \dots$ where α are roots of $b(\lambda)$.

We will now reformulate Theorem 3 in terms of modules over the algebra $\mathcal{D}$. More precisely, let $\tilde{\mathcal{D}} = \mathcal{D} \otimes \mathbb{C}(\lambda)$, algebra over the field $\mathbb{C}(\lambda)$. Let $M(p)$ be the $\tilde{\mathcal{D}}$-module spanned by formal expressions $q \cdot p^{\lambda-i}$, $i \in \mathbb{Z}$, $q \in \mathbb{C}(\lambda)[x_1, \dots, x_n]$, with relations $q p^{\lambda-i+1} = qp \cdot p^{\lambda-i}$ (i.e. $M(p) = \mathbb{C}(\lambda)[x_1, \dots, x_n][\frac{1}{p}] \cdot p^\lambda$). This is indeed a $\tilde{\mathcal{D}}$-module, with the action of $\tilde{\mathcal{D}}$ defined in a natural way.

<u>Theorem 4.</u> $M(p)$ is a finitely generated $\tilde{\mathcal{D}}$-module.

Let us show that Theorem 4 implies Theorem 3 (in fact, they are equivalent)

Let M_i be the submodule of $M(p)$ generated by $p^{\lambda-i}$. Then $M_i \subset M_{i+1}$, and $M(p) = \bigcup_{i\in\mathbb{Z}} M_i$. Assume that $M(p)$ is finitely generated. This implies that there is j such that $M(p) = M_j$.

In other words, $M(p)$ is generated by $p^{\lambda-j}$, and thus there is $\tilde{L} \in \tilde{\mathcal{D}}$ such that

$p^{\lambda-j-1} = \tilde{L}\, p^{\lambda-j}$

Let $\sigma : \mathbb{C}(\lambda) \to \mathbb{C}(\lambda)$, $\sigma(\lambda) = \lambda+j+1$

Let $\sigma(\tilde{L}) = \dfrac{L}{b(\lambda)}$, where $L \in \mathcal{D}[\lambda]$.

Then $\dfrac{L}{b(\lambda)} p^{\lambda+1} = p^{\lambda}$, so $Lp^{\lambda+1} = b(\lambda) p^{\lambda}$, and Theorem 3 follows.

We now want to prove Theorem 4. For this we'll need some machinery.

If V is a vector space ~~an associative algebra~~, by an increasing filtration we will mean a collection of subspaces $F_iV \subset V$, $i \geq 0$ such that

1) $F_0V \subset F_1V \subset \cdots$

2) $\bigcup_i F_iV = V$

We will agree that $F_{-1}V = 0$ and $F_{-i}V = 0$ for $i < -1$

If $V=A$ is an associative algebra, we'll require that $1 \in F_0 A$ and $F_i A \cdot F_j A \subseteq F_{i+j} A$.

Associated graded space:

$$gr^F V = \bigoplus_{i=0}^{\infty} \underbrace{F_{i+1} V / F_i V}_{\substack{\| \\ gr_i^F V}}.$$

If $V=A$ is an algebra, then so is $gr^F A$. (associated graded algebra). We have $gr_i^F A \cdot gr_j^F A \subseteq gr_{i+j}^F A$

If M is a left module over a filtered algebra A, then an increasing filtration of M is a filtration on M as a vector space, such that $F_i A \cdot F_j M \subseteq F_{i+j} M$.

Then $gr^F M$ is a graded $gr^F A$-module:

$gr_i^F A \cdot gr_j^F M \subseteq gr_{i+j}^F M$.

<u>Def.</u> (1) A Filtration $F_\circ$ on M is called <u>good</u> if $gr^F M$ is a finitely generated $gr^F A$-module.

(2) Filtrations F and F' on M are called equivalent if $\exists j_0, j_1 \in \mathbb{Z}_+$ such that

$$F'_{j-j_0} M \subseteq F_j M \subseteq F'_{j+j_1} M.$$

Proposition 5 (1) Let F be a good filtration on M, and $\overline{m}_1, \dots, \overline{m}_n$ be generators of $gr^F M$ over $gr A$ of degrees $d_1, \dots, d_n$. Let $m_1, \dots, m_n$ be any lifts of $\overline{m}_1, \dots, \overline{m}_n$. Then for any $j \geq 0$,

$$F_j M = F_{j-d_1} A \cdot m_1 + \dots + F_{j-d_n} A \cdot m_n.$$

In particular, M is a finitely generated A-module with generators $m_1, \dots, m_n$.

(2) If F' is another filtration on M then $\exists k$ such that $\forall j$ $F_j M \subseteq F'_{j+k} M$.

Proof. (1) We have $gr_j^F M = gr^F_{j-d_1} A \cdot \overline{m}_1 + \dots + gr^F_{j-d_n} A \cdot \overline{m}_n$, so the statement follows by induction in j.

(2) Let k be such that $m_i \in F'_k M$ $\forall i$. Then by (1) $F_j M \subseteq F'_{j+k} M$ $\forall j$.

Corollary 6. If F, F' are good filtrations on M then they are equivalent.

Main example: $\mathcal{D}$ is the algebra of polynomial differential operators in n variables over a field k. There are two important filtrations on $\mathcal{D}$:

1) Bernstein filtration: $F_i \mathcal{D}$ is the span of all monomials in degree $\leq i$ in x_i and ∂_i, i.e. image of $F_1 \mathcal{D}^{\otimes i}$ under mult

This is defined by
$dy(x_j) = dy(\partial_j) = 1$.

2) Geometric filtration (filtration by order of differential operator)

$$\mathcal{D}_0 \subset \mathcal{D}_1 \subset \cdots \qquad \mathcal{D} = \bigcup_{i \geq 0} \mathcal{D}_i.$$

$\mathcal{D}_i$ = span of all monomials in x_j, ∂_j containing $\leq i$ derivatives.

Lemma 7 (easy). For any $L \in \mathcal{D}$ there is a unique representation

$$L = \sum_{1 \leq i_1 \leq \dots \leq i_k \leq n} P_{i_1 \dots i_k}(x_1, \dots, x_n)\, \partial_{i_1} \cdots \partial_{i_k}$$

Corollary 8. For both filtrations, $\mathrm{gr}\,\mathcal{D} = \mathbb{C}[x_1, \dots, x_n, \xi_1, \dots, \xi_n]$, where ξ_i are the images of ∂_i.

Proof. We need to show that $[\xi_j, x_i] = 0$ ~~[...]~~. This is clear, since $[\partial_j, x_i] = \delta_{ij}$, element of lower degree.
So $[\mathcal{D}_i, \mathcal{D}_j] \subseteq \mathcal{D}_{i+j-1}$.

Remark. In fact, for Bernstein filtration we get a little more: $[F_i\mathcal{D}, F_j\mathcal{D}] \subseteq F_{i+j-2}\mathcal{D}$.

Prop 9. Any f.g. A-module M has a good filtration.

Proof. Let $m_1, \dots, m_n$ be generators of M, and let $F_iM = F_iA \cdot m_1 + \dots + F_iA \cdot m_n$. Then the images $\bar{m}_i$ of m_i in $\mathrm{gr}^F_0 M$ are generators of $\mathrm{gr}M$ over $\mathrm{gr}A$. ∎

Ex. $A = \mathbb{C}[t]$, $M = A$, $F_0M = F_1M = \dots = M$. Then $\mathrm{gr}M = M$, and $t|_M = 0$. So this is not a good filtration.