

Fall 2013, Course 18.769: Algebraic D-modules

TR 11-12.30, Rm. E17-128

Instructor: Pavel Etingof

Office: E17-430

Telephone: 253-3669

Email: etingof@math.mit.edu

Office hours: Tuesday 4-5, Friday 2-3

From classical analysis we know that

$$\int_{-\infty}^{\infty} |x|^s e^{-x^2} dx = \Gamma\left(\frac{s+1}{2}\right),$$

where Γ is the Euler Gamma function. This means that the integral on the left hand side, initially defined and analytic only if $\operatorname{Re}(s) > -1$, in fact extends to a meromorphic function of s in the whole complex plane, which has poles at the points $-1, -3, -5, \dots$. It turns out that a similar statement holds much more generally, in several variables, when we don't have any explicit formula for the integral. Namely, let P be a polynomial of n variables $x = (x_1, \dots, x_n)$, and ϕ a rapidly decaying test function of these variables. Then one can show that the integral

$$\int_{\mathbf{R}^n} |P(x)|^s \phi(x) dx,$$

initially defined and analytic for $\operatorname{Re}(s) > 0$, extends meromorphically to the whole plane, with poles on finitely many one-sided arithmetic progressions, of the form $-a, -a-1, -a-2, \dots$, $\operatorname{Re}(a) > 0$. This seemingly analytic fact has algebraic roots, and one of its proofs, due to Joseph Bernstein, is based on the properties of modules over the ring of polynomial differential operators in several variables, which are called (algebraic) D-modules (on the affine space).

We will first develop the theory of D-modules on the affine space, and prove Bernstein's theorem using this theory. An important role in this theory will be played by Bernstein's inequality, which is an algebraic incarnation of the Heisenberg uncertainty principle in quantum mechanics. Then we will generalize this theory to the case of D-modules on algebraic varieties, by considering modules over the algebra $D(X)$ of differential operators on a smooth variety X . The algebra $D(X)$ is a quantum (noncommutative) deformation of the algebra of functions on the symplectic variety T^*X , so the theory of D-modules can be viewed as a kind of noncommutative geometry. In particular, we will study the important notion of a holonomic D-module, which generalizes the notion of a holonomic system of differential equations (vector bundle with a flat connection), and is a quantization of the notion of a Lagrangian subvariety in T^*X .

We will see that the categories of D-modules on algebraic varieties have a rich structure. Namely, there are functors of direct and inverse image between them (ordinary and with compact support) associated to morphisms of algebraic varieties (some of which are defined only on holonomic D-modules), as well as functors of duality and external tensor product. They satisfy many nice properties (formalism of six functors). The only caveat is that these functors not only fail to be exact, but in general are not even defined on the abelian categories of D-modules, but are only defined on their derived categories. So as a part of the course we will have to develop the basic theory of triangulated and derived categories, before we fully develop the formalism of six functors.

Next, we will talk about D-modules with regular singularities, which are generalizations of Fuchsian differential equations. We will discuss the Riemann-Hilbert correspondence for bundles with a flat connection and regular singularities, and how to generalize this correspondence to D-modules with regular singularities. This will lead us to the notion of perverse sheaves.

Finally, we will discuss applications of D-modules to representation theory, namely the Beilinson-Bernstein localization theorem.

While the course contains many abstract notions and constructions, the style of the course will be to try to understand them through working with concrete examples (usually coming from analysis and differential equations), which will also provide motivation for these constructions. In other words, we will try, as much as possible, not to lose the connection with classical analysis, which originally motivated the development of this subject.

I am not planning to use any particular book as the main source (although see a list of books on D-modules below). I will write my own notes, which will be made available, and use some notes available online:

"Lectures on algebraic D-modules" by A. Braverman and T. Chmutova,

http://www.math.harvard.edu/~gaitsgde/grad_2009/Dmod.brav.pdf

"Lectures on D-modules" by V. Ginzburg

http://www.math.harvard.edu/~gaitsgde/grad_2009/Ginzburg.pdf

"Lectures on the algebraic theory of D-modules" by D. Milicic

<http://www.math.utah.edu/~milicic/Eprints/dmodules.pdf>

"Algebraic theory of D-modules" by J. Bernstein,

<http://www.math.columbia.edu/~khovanov/resources/Bernstein-dmod.pdf>

The background needed for this course is basic knowledge of commutative algebra and algebraic geometry, as well as basic homological algebra.

To officially pass the course for students who have not passed the qualifying exam, it will be required to solve homework assignments which will be assigned on Tuesdays and due the following Tuesday.

References

Borel, Armand, ed. (1987), Algebraic D-Modules, Perspectives in Mathematics 2, Boston, MA: Academic Press

Coutinho, S. C. (1995), A primer of algebraic D-modules, London Mathematical Society Student Texts 33, Cambridge University Press

Kashiwara M. (2003), D-modules and microlocal calculus, AMS.