

18.747: Problem set 9; due Thursday, April 17

1. (a) Check that the positive part \mathfrak{n}_+ of Lie algebra $sl(3)$ is defined by the standard generators e_i and the Serre relations.

(b) Do the same for $sp(4)$.

Hint: Try to write down the basis of the Lie algebra generated by e_i and the Serre relations, and compare its cardinality with $\dim(\mathfrak{n}_+)$.

2. Establish Serre relations for the affine Lie algebra which is the 1-dimensional central extension of the loop algebra $L\mathfrak{g}$ of a simple Lie algebra \mathfrak{g} .

3. The vertex operator construction. Let a_i be the standard generators of the Heisenberg algebra. Let F_μ be the Fock space of this algebra (where a_0 acts by μ), and $F = \bigoplus_{m \in \mathbb{Z}} F_{\sqrt{2}m}$. Define Vertex operators on F :

$$X_\pm(u) = \exp(\mp\sqrt{2} \sum_{n<0} \frac{a_n}{n} u^{-n}) \exp(\mp\sqrt{2} \sum_{n>0} \frac{a_n}{n} u^{-n}) S^{\pm 1} u^{\pm\sqrt{2}a_0},$$

where S is the operator of shift $m \rightarrow m + 1$ (this is really the same as Γ, Γ^* in Kac-Raina, except for factors of $\sqrt{2}$).

(a) Show that

$$X_a(u)X_b(v) = (u-v)^{2ab} : X_a(u)X_b(v) :, a, b = \pm 1.$$

(this is again the same as in Kac-Raina, except for the factor of 2). In particular,

$$X_a(u)X_b(v) = X_b(v)X_a(u),$$

in the sense that matrix elements of both are series which converge (in different regions) to the same rational functions. (note that in the situation of Kac-Raina there was a minus sign; thus while Γ, Γ^* were fermions, X_\pm are bosons).

(b) Calculate $\langle 1, X_+(u_1) \dots X_+(u_n) X_-(v_1) \dots X_-(v_n) 1 \rangle$, where 1 is the highest weight vector of F_0 .

(c) Find the commutation relation of X_\pm with a_i .

4. (a) Define $e(u) = X_+(u)$, $f(u) = X_-(u)$, $h(u) = \sqrt{2}a(u) = \sqrt{2} \sum a_i u^{-i-1}$. Show that the components of these power series are operators on F which define a representation of the affine Lie algebra $\widehat{sl(2)}$. Show that this representation is highest weight, and that it has level one and highest weight 0 (with respect to $sl(2)$).

(b) Show that the representation of the affine algebra in F is irreducible. Compute its character and compare with Kac character formula.