18.747: Problem set 8; due Thursday, April 24

1. (a) Check that the positive part $n_+$ of Lie algebra $sl(3)$ is defined by the standard generators $e_i$ and the Serre relations.

(b) Do the same for $sp(4)$.

Hint: Try to write down the basis of the Lie algebra generated by $e_i$ and the Serre relations, and compare its cardinality with $\dim(n_+)$. 

2. Establish Serre relations for the affine Lie algebra which is the 1-dimensional central extension of the loop algebra $Lg$ of a finite dimensional simple Lie algebra $g$.

3. The vertex operator construction. Let $a_i$ be the standard generators of the Heisenberg algebra. Let $F_\mu$ be the Fock space of this algebra (where $a_0$ acts by $\mu$), and $F = \oplus_{m \in \mathbb{Z}} F_{\sqrt{2}m}$. Define Vertex operators on $F$:

$$X_\pm(u) = \exp(\mp \sqrt{2} \sum_{n<0} \frac{a_n}{n} u^{-n}) \exp(\mp \sqrt{2} \sum_{n>0} \frac{a_n}{n} u^{-n}) S^{\pm 1} \sqrt{2} \sum_{n=0}^\infty $$

where $S$ is the operator of shift $m \rightarrow m + 1$ (this is really the same as $\Gamma, \Gamma^*$ in Kac-Raina, except for factors of $\sqrt{2}$).

(a) Show that

$$X_a(u)X_b(v) = (u - v)^{2ab} : X_a(u)X_b(v) ; : a, b = \pm 1.$$  

(this is again the same as in Kac-Raina, except for the factor of 2). In particular,

$$X_a(u)X_b(v) = X_b(v)X_a(u),$$

in the sense that matrix elements of both are series which converge (in different regions) to the same rational functions. (note that in the situation of Kac-Raina there was a minus sign; thus while $\Gamma, \Gamma^*$ were fermions, $X_\pm$ are bosons).

(b) Calculate $< 1, X_+(u_1)...X_+(u_n)X_-(v_1)...X_-(v_n)1 >$, where 1 is the highest weight vector of $F_0$.

(c) Find the commutation relation of $X_\pm$ with $a_i$.

4. (a) Define $e(u) = X_+(u)$, $f(u) = X_-(u)$, $h(u) = \sqrt{2}u = \sqrt{2} \sum a_i u^{-i-1}$. Show that the components of these power series are operators on $F$ which define a representation of the affine Lie algebra $\widehat{sl(2)}$. Show that this representation is highest weight, and that it has level one and highest weight 0 (with respect to $sl(2)$).

(b) Show that the representation of the affine algebra in $F$ is irreducible. Compute its character, i.e $\text{Tr}_F(e^{n^h}q^d)$, where $d$ is the degree operator defined by the condition that it annihilates the highest weight vector, and $[d, at^\mu] = nat^\mu$, $a \in sl(2)$. 

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