18.747: Problem set 9; due Thursday, November 7

1. (a) Check that the positive part $\mathfrak{n}_+$ of Lie algebra $sl(3)$ is defined by the standard generators e_i and the Serre relations.
(b) Do the same for $sp(4)$.
Hint: Try to write down the basis of the Lie algebra generated by e_i and the Serre relations, and compare its cardinality with $\dim(\mathfrak{n}_+)$.
2. Establish Serre relations for the affine Lie algebra which is the 1-dimensional central extension of the loop algebra $L\mathfrak{g}$ of a finite dimensional simple Lie algebra $\mathfrak{g}$.
3. **The vertex operator construction.** Let a_i be the standard generators of the Heisenberg algebra. Let F_μ be the Fock space of this algebra (where a_0 acts by μ), and $F = \oplus_{m\in Z} F_{\sqrt{2}m}$. Define Vertex operators on F:

$$X_\pm(u) = \exp(\mp\sqrt{2}\sum_{n<0}\frac{a_n}{n}u^{-n})\exp(\mp\sqrt{2}\sum_{n>0}\frac{a_n}{n}u^{-n})S^{\pm 1}u^{\pm\sqrt{2}a_0},$$

where S is the operator of shift $m \to m+1$ (this is really the same as Γ, Γ^* in Kac-Raina, except for factors of $\sqrt{2}$).
(a) Show that

$$X_a(u)X_b(v) = (u-v)^{2ab} : X_a(u)X_b(v) :, a, b = \pm 1.$$

(this is again the same as in Kac-Raina, except for the factor of 2). In particular,

$$X_a(u)X_b(v) = X_b(v)X_a(u),$$

in the sense that matrix elements of both are series which converge (in different regions) to the same rational functions. (note that in the situation of Kac-Raina there was a minus sign; thus while Γ, Γ^* were fermions, $X_\pm$ are bosons).
(b) Calculate $< 1, X_+(u_1)...X_+(u_n)X_-(v_1)...X_-(v_n)1 >$, where 1 is the highest weight vector of F_0.
(c) Find the commutation relation of $X_\pm$ with a_i.
4. (a) Define $e(u) = X_+(u)$, $f(u) = X_-(u)$, $h(u) = \sqrt{2}a(u) = \sqrt{2}\sum a_i u^{-i-1}$. Show that the components of these power series are operators on F which define a representation of the affine Lie algebra $\widehat{sl(2)}$. Show that this representation is highest weight, and that it has level one and highest weight 0 (with respect to $sl(2)$).
(b) Show that the representation of the affine algebra in F is irreducible. Compute its character, i.e $\mathrm{Tr}_F(e^{zh}q^d)$, where d is the degree operator defined by the condition that it annihilates the highest weight vector, and $[d, at^n] = nat^n$, $a \in sl(2)$.