18.747: Problem set 8; due Thursday, October 31

1. The Weyl-Kac denominator formula for affine Lie algebras states that

$$\sum_{w \in W} \epsilon(w) e^{(w\rho - \rho, h)} = \prod_{\gamma} (1 - e^{-(\gamma, h)})^{\text{mult}(\gamma)},$$

where the sum is taken over positive roots (see Remark 11.3 in Kac-Raina). Derive it in the case of affine sl(2) from the Jacobi triple product identity.

2. Let $M_\lambda^+$ be the highest weight Verma module over a finite dimensional simple Lie algebra $g$, and $M_\mu^-$ be the lowest weight Verma module. Let $V$ be any $h$-diagonalizable module over this Lie algebra. Show that $\text{Hom}(M_\lambda^+ \otimes M_\mu^-, V)$ (as representations) is isomorphic to the space $V[\lambda + \mu]$ of vectors in $V$ of weight $\lambda + \mu$.

3. Let $\Theta_{n,m}(z, \tau)$ be the function defined by (11.15) in Kac-Raina (for $u = 0$). Fix $\tau$ in the upper half of the complex plane.
   (a) Show that for fixed $\tau$ this is a holomorphic function of $z$ for all $z$.
   (b) Relate $\Theta_{n,m}$ with $\Theta_{0,1}$.
   (c) Find the zeros of $\Theta_{n,m}$ and their multiplicities. Hint. Use Jacobi triple product to factorize $\Theta_{0,1}$. 

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