18.747: Problem set 8; due Thursday, October 31

1. The Weyl-Kac denominator formula for affine Lie algebras states that

$$\sum_{w\in W} \epsilon(w) e^{(w\rho-\rho,h)} = \prod_{\gamma} (1 - e^{-(\gamma,h)})^{mult(\gamma)},$$

where the sum is taken over positive roots (see Remark 11.3 in Kac-Raina). Derive it in the case of affine sl(2) from the Jacobi triple product identity.
2. Let M_λ^+ be the highest weight Verma module over a finite dimensional simple Lie algebra g, and M_λ^- be the lowest weight Verma module. Let V be any h-diagonalizable module over this Lie algebra. Show that $Hom(M_\lambda^+ \otimes M_\mu^-, V)$ (as representations) is isomorphic to the space $V[\lambda+\mu]$ of vectors in V of weight $\lambda+\mu$.
3. Let $\Theta_{n,m}(z,\tau)$ be the function defined by (11.15) in Kac-Raina (for $u=0$). Fix τ in the upper half of the complex plane.
(a) Show that for fixed τ this is a holomorphic function of z for all z.
(b) Relate $\Theta_{n,m}$ with $\Theta_{0,1}$.
(c) Find the zeros of $\Theta_{n,m}$ and their multiplicities. Hint. Use Jacobi triple product to factorize $\Theta_{0,1}$.