

18.747: Problem set 8; due Thursday, April 10

1. The Weyl-Kac denominator formula for affine Lie algebras states that

$$\sum_{w \in W} \epsilon(w) e^{(w\rho, h)} = \prod_{\gamma} (1 - e^{-(\gamma, h)})^{\text{mult}(\gamma)},$$

where the sum is taken over positive roots (see Remark 11.3 in Kac-Raina). Derive it in the case of affine $\mathfrak{sl}(2)$ from the Jacobi triple product identity.

2. Let M_{λ}^{+} be the highest weight Verma module over a finite dimensional simple Lie algebra \mathfrak{g} , and M_{λ}^{-} be the lowest weight Verma module. Let V be any h -diagonalizable module over this Lie algebra. Show that $\text{Hom}(M_{\lambda}^{+} \otimes M_{\mu}^{-}, V)$ (as representations) is isomorphic to the space $V[\lambda + \mu]$ of vectors in V of weight $\lambda + \mu$.

3. Let $\Theta_{n,m}(z, \tau)$ be the function defined by (11.15) in Kac-Raina (for $u = 0$). Fix τ in the upper half of the complex plane.

- (a) Show that for fixed τ this is a holomorphic function of z for all z .
- (b) Relate $\Theta_{n,m}$ with $\Theta_{0,1}$.
- (c) Find the zeros of $\Theta_{n,m}$ and their multiplicities. Hint. Use Jacobi triple product to factorize $\Theta_{0,1}$.