18.747: Problem set 7; due Thursday, October 24

1. Let $L\mathfrak{g} = \mathfrak{g}[t, t^{-1}]$ be the loop algebra of a simple finite dimensional Lie algebra $\mathfrak{g}$. Let z be a nonzero complex number and V be a finite dimensional representation of $\mathfrak{g}$. Define a representation $V(z)$ of $L\mathfrak{g}$ by $a(n)|_V = z^n a|_V$, where $a(n) = at^n$. This is called an evaluation representation.

(a) Consider the representation $V_1(z_1) \otimes ... \otimes V_n(z_n)$, where V_i are irreducible nontrivial representations of $\mathfrak{g}$. Show that this representation is irreducible if and only if z_i are distinct.

Hint. You will have to show that if V, W are irreducible nontrivial representations of $\mathfrak{g}$ then $V \otimes W$ is reducible. To this end, consider the endomorphism algebra $\mathrm{End}_{\mathfrak{g}}(V \otimes W)$ of this representation over $\mathfrak{g}$, and show, by using dual representations, that it is at least two-dimensional (use that $V \otimes V^*$ and $W \otimes W^*$ contain copies of $\mathbf{C}$ and $\mathfrak{g}$).

(b) When are two such irreducible representations isomorphic?

Hint. Consider the eigenvalues of $h(n)$, where h is in the Cartan subalgebra.

2. (a) Show that any irreducible finite dimensional representation of $L\mathfrak{g}$ (where $\mathfrak{g}$ is as above) has the form as in problem 1 (the empty product should be interpreted as the trivial representation).

Hint. Let V be an irreducible finite dimensional representation of $L\mathfrak{g}$, and let I be the kernel of the map $\phi : \mathbf{C}[t, t^{-1}] \to \mathrm{Hom}_{\mathbf{C}}(\mathfrak{g}, \mathrm{End}(V))$ given by $\phi(f)(a)v = (a \otimes f)v$, $a \in \mathfrak{g}$, $v \in V$. Show that I is an ideal of finite codimension. Deduce that the action of $L\mathfrak{g}$ on V factors through the finite dimensional Lie algebra $\mathfrak{g} \otimes \mathbf{C}[t, t^{-1}]/I$. Then classify such V, using the primary decomposition of I and Lie's theorem.

(b) Let R be a commutative $\mathbf{C}$-algebra. Classify irreducible finite dimensional complex representations of the Lie algebra $\mathfrak{g} \otimes R$ in terms of maximal ideals of R.

3. Can you give an example of a reducible but not decomposable finite dimensional representation of $L\mathfrak{g}$?