18.747: Problem set 6; due Thursday, October 17

1. Prove formula 9.14b in Kac-Raina.
2. Let $\mathcal{F} = \oplus \mathcal{F}^{(m)}$ be the semiinfinite wedge space.
(a) Show that $\mathcal{F}$ is an irreducible representation of the Clifford algebra generated by $\hat{v}_j, \check{v}_j^*$.
(b) Compute $Tr_{\mathcal{F}}(q^{\mathbf{d}} z^{\mathbf{m}})$, where $\mathbf{d}$ is the operator multiplying homogeneous elements by their degree, defined by $deg(\psi_0) = 0$, $deg(\hat{v}_j) = j$, $deg(\check{v}_j^*) = -j$, and $\mathbf{m}$ is the operator which acts by multiplication by the number m on $\mathcal{F}^{(m)}$.
3. Using the fact that $\mathcal{F} = \mathcal{B}$, compute the answer to problem 2 using the bosonic realization. Deduce that

$$\prod_{n \geq 0} (1 - q^n z)(1 - q^{n+1} z^{-1})(1 - q^{n+1}) = \sum_{m \in Z} (-z)^m q^{m(m-1)/2}$$

(Jacobi triple product identity). Substitute $z = q^{1/3}$ and obtain Euler's pentagonal identity for $\prod_{n \geq 1} (1 - p^n)$.
4. Let $\hat{\mathfrak{g}}$ be the affine Lie algebra associated to a simple Lie algebra $\mathfrak{g}$. Let $a \in \mathfrak{g}$ and $a(z) = \sum a[n] z^{-n-1}$, $a[n] = at^n \in \hat{\mathfrak{g}}$.
(a) Show that if V is a highest weight representation of $\hat{\mathfrak{g}}$ then $a(z)$ defines a linear map $V \to V((z))$ (here $V((z))$ is the space of formal Laurent series with coefficients in V).
(b) Let V have a highest weight vector v with $hv = 0$ for h in the Cartan subalgebra of $\mathfrak{g}$, and $Kv = kv$. Compute $< v, a(z_1) b(z_2) v >$ (as a rational function).
(c) Compute $< v, a(z_1) b(z_2) c(z_3) v >$.