18.747: Problem set 5; due Tuesday, March 8

1. (optional; for those who like calculations!) Prove identity (7.18) in Kac-Raina.

2. Let $B^{(0)}$ be the Fock space and $a(z)$ be the basic quantum field ($a(z) = \sum a_n z^{-n-1}$). Let $a(z_1)...a(z_n)$ be the usual product in which the annihilation operators $a_n$ $n > 0$, have been moved to the right (we agree that this is 1 if $n = 0$).

(a) Express $a(z_1)...a(z_n)$ as a linear combination of $:a(z_j)...a(z_{j'}):$ with coefficients depending on $z_j$ (e.g. $a(z_1)a(z_2) = a(z_1)a(z_2) : + 1/(z_1 - z_2)^2$).

(b) Prove the formula for $<1^*, a(z_1)...a(z_{2n})1>$ from a previous homework using (a).

3. Let $d$ be the degree operator in Fock space $B^{(0)}$ (i.e. it multiplies a homogeneous vector by its degree, where $\text{deg}(x_i) = i$). Let $\Gamma(u, v)$ be the operator on this space introduced in Kac-Raina. Show that

$$\text{tr}(\Gamma(u, v) q^d) = \prod_{n \geq 1} \frac{1 - q^n}{(1 - q^nu/v)(1 - q^nv/u)}$$

(as formal series).

Hint: Compute the trace of the operator $e^{az} e^{b\partial} q^{x0}$ in the space of polynomials in one variable, then obtain the answer for infinitely many variables by tensoring and algebraic manipulations.