18.747: Problem set 5; due Thursday, October 10

1. (optional; for those who like calculations!) Prove identity (7.18) in Kac-Raina.

2. Let $\mathcal{B}^{(0)}$ be the Fock space and $a(z)$ be the basic quantum field ($a(z) = \sum a_n z^{-n-1}$). Let $: a(z_1)...a(z_n) :$ be the usual product in which the annihilation operators a_n $n > 0$, have been moved to the right (we agree that this is 1 if $n = 0$).

(a) Express $a(z_1)...a(z_n)$ as a linear combination of $: a(z_{j^1})...a(z_{j^r}) :$ with coefficients depending on z_j (e.g. $a(z_1)a(z_2) =: a(z_1)a(z_2) : +1/(z_1 - z_2)^2$).

(b) Prove the formula for $< 1^*, a(z_1)...a(z_{2n})1 >$ from a previous homework using (a).

3. Let d be the degree operator in Fock space $\mathcal{B}^{(0)}$ (i.e. it multiplies a homogeneous vector by its degree, where $\deg(x_i) = i$). Let $\Gamma(u, v)$ be the operator on this space introduced in Kac-Raina. Show that

$$tr(\Gamma(u, v)q^d) = \prod_{n \geq 1} \frac{1 - q^n}{(1 - q^n u/v)(1 - q^n v/u)}$$

(as formal series).

Hint: Compute the trace of the operator $e^{ax} e^{b\partial} q^{x\partial}$ in the space of polynomials in one variable, then obtain the answer for infinitely many variables by tensoring and algebraic manipulations.