18.747: Problem set 5; due Thursday, March 13

1. Prove identity (7.18) in Kac-Raina.
2. Let $\mathcal{B}^{(0)}$ be the Fock space and $a(z)$ be the basic quantum field ($a(z) = \sum a_n z^{-n-1}$). Let $:a(z_1)...a(z_n):$ be the usual product in which the annihilation operators a_n $n > 0$, have been moved to the right (we agree that this is 1 if $n = 0$).
(a) Express $a(z_1)...a(z_n)$ as a linear combination of $:a(z_{j^1})...a(z_{j^r}):$ with coefficients depending on z_j (e.g. $a(z_1)a(z_2) =: a(z_1)a(z_2): +1/(z_1 - z_2)^2$).
(b) Prove the formula for $< 1^*, a(z_1)...a(z_{2n})1 >$ using (a).
3. Let d be the degree operator in Fock space $\mathcal{B}^{(0)}$ (i.e. it multiplies a homogeneous vector by its degree). Let $\Gamma(u, v)$ be the operator on this space introduced in Kac-Raina. Show that

$$tr(\Gamma(u,v)q^d) = \prod_{n\geq 1} \frac{1-q^n}{(1-q^n u/v)(1-q^n v/u)}$$

(as formal series).
Hint: Compute the trace of the operator $e^{ax}e^{b\partial}q^{x\partial}$ in the space of polynomials in one variable, then obtain the answer for infinitely many variables by tensoring and algebraic manipulations.