

18.747: Problem set 4; due Thurssday, March 6

1. Show that in problem 3 of Set 2,

$$\langle 1^*, a(z_1) \dots a(z_{2n}) 1 \rangle = \sum_{\{s \in S_{2n}; s^2 = 1, s(i) \neq i \forall i\}} \prod_{i < s(i)} \frac{1}{(z_i - z_{s(i)})^2},$$

where the right hand side means the expansion in the region $|z_1| > \dots > |z_{2n}|$.

2. Let 1 denote the highest weight vector of the bosonic space $\mathcal{B}^{(0)}$, and \langle, \rangle denote the contravariant form on $\mathcal{B}^{(0)}$.

(a) Compute the inner product $\langle 1, \Gamma(u_1) \dots \Gamma(u_n) \Gamma^*(v_1) \dots \Gamma^*(v_n) 1 \rangle$, by using the explicit form of $\Gamma(u)$ (as a vertex operator). You'll have to use the formula on p.56 of Kac-Raina, after (5.31b).

(b) Compute the same inner product using the fermionic realization of $\mathcal{B}^{(0)}$, i.e. replacing Γ, Γ^* with X, X^* (this should be analogous to problem 1).

By equating the results of (a),(b), deduce the identity

$$\frac{\prod_{i < j} (u_i - u_j) \prod_{i < j} (v_i - v_j)}{\prod_{i, j} (u_i - v_j)} = (-1)^{n(n-1)/2} \det \left(\frac{1}{u_i - v_j} \right)$$

This identity can be regarded as a “numerical” version of the Boson-Fermion correspondence.

(c) Give an elementary proof of the identity from (b) (Multiply both sides by the denominator of the left hand side, then show that the polynomial on the RHS is divisible by $u_i - u_j$ and $v_i - v_j$, and that it has the smallest possible degree among such polynomials. Conclude that the two sides of the equality are proportional, and show that the proportionality coefficient is 1 by considering asymptotics at infinity).